Audience Design

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11th June 2024

Abstract

Does a sender benefit from communicating with an audience in groups rather than publicly or privately? In a cheap-talk game, I show that the sender can gain credibility by ensuring diversity of opinions in a group so that her incentive to lie to a subset of the group is offset by her incentive to be truthful to the rest. The sender's optimal grouping, or partition, of the audience maximises her benefit from gaining credibility from each group. Public or private communications are not necessarily optimal when the sender can benefit from differently diverse groups of receivers. When the sender values each receiver equally and can gain credibility only by ensuring diversity of opinions in her audience, I show that it is optimal for the sender to separate those who need to be persuaded from those who do not. I also derive further properties of optimal communication when receivers are "single-minded," and demonstrate the role of diversity in shaping optimal communication.

A politician is trying to persuade a voter by claiming that hers and the voter's interests are aligned. Would the voter find such an argument persuasive? A valid concern for the voter is that the politician could be making the same claim to other voters with conflicting interests. Such a concern might mean that the voter would not find the politician's argument persuasive. But what if the politician was making the same claim in the presence of another voter with conflicting interests? The politician's claim that her interest is aligned with that of the original voter is then also a statement that her and the other voter's interests are not aligned. Thus, the same statement—that "our interests are aligned"—is more credible

*The Department of Economics, Cornell University, takumahabu@cornell.edu. The paper was previously circulated under the title "The Power of Semi-Public Communication." I thank my committee members Emir Kamenica, Elliot Lipnowski, Doron Ravid and Lars Stole for their invaluable advice and feedback. I am also grateful to Marco Battaglini, Ben Brooks, Andy Choi, Stephen Coate, Alex Frankel, Mike Gibbs, Alejandro Martínez-Marquina, Canice Prendergast, Phil Reny, Christoph Schlom, Nancy Stokey, Thomas Winberry, Kai Hao Yang, and the participants of the third-year research class and the micro-theory seminar at The University of Chicago for their helpful comments.

in the presence of the other voter because the claim now has stakes for the politician.[1] This intuition might suggest that the politician would benefit from speaking publicly in front of all the voters. In this paper, I demonstrate that the politician is often better off communicating to voters in multiple groups (by, for example, inviting different sets of voters to different events) because doing so allows her to adopt different arguments that are most persuasive to each group that are differently diverse.

More concretely, I study a cheap-talk game in which a sender wishes to persuade a heterogeneous set of receivers to each always take a particular individual action, and each receiver's action is equally valuable for the sender.[2] The departure from existing literature is that I allow the sender to communicate *semi-publicly* with the receivers; i.e., the sender can form a partition of receivers and choose a messaging strategy that specifies state-contingent message for each group of the partition.[3] As in the motivating example above, communicating to a diverse group of receivers can have a disciplining effect on the sender's incentive to lie and thus lend credibility to the sender's cheap-talk communication. However, not all kinds of diversities are equal because the presence of some voters may not, in fact, have the desired disciplining effect. Thus, communicating semi-publicly—as opposed to publicly—involves the sender ensuring that each group contains an appropriately diverse set of receivers to make her communication to each group credible.

Toward characterising the sender's optimal semi-public communication, I first show that the sender's messaging strategy with respect to each group can be independent across groups (Lemma 1). Hence, the sender's payoff from any partition can be expressed as a sum of her payoffs from each group, where her payoff from a group is taken to be her preferred equilibrium payoff of a public cheap-talk game with respect to the group. Consequently, the sender's problem is to maximise the sum of payoffs from each group over the set of all possible partitions of receivers; i.e., the sender's problem is an example of a *Coalition Structure Generation (CSG) problem* that is well-studied—but is known to be computationally difficult to solve—in the computer science literature. I then show that it is always optimal for the sender to separate the audience into at least two groups (Theorem 1): one group consisting of those that do not need persuading (i.e., receivers who take the

[1] Farrell and Gibbons (1989) first observed this effect in the context of a sender-receiver game with two receivers.

[2] As I discuss later, the sender's transparent motive to persuade receivers implies that diversity in an audience is the only potential source of credibility for the sender.

[3] Public and private communication are special cases of semi-public communication in which the partition of the audience is either the entire audience or the collection of singleton sets of individuals in the audience, respectively.

sender-preferred action without any information) and the other consisting of those that do need persuading. This result implies that, for example, there is no need to ensure diversity of opinions in political rallies that are held for the supporters. The result also means that the sender's problem of finding optimal communication is nontrivial only for the latter group of receivers consisting of those that do need persuading.

To make progress, I specialise the model by assuming that receivers are *single-minded*; i.e., each receiver takes the sender-preferred action if and only if their belief that the state is their preferred one is above a threshold.[4] For example, if we take the unknown state as the single political issue that the politician cares about, a single-minded receiver is a voter who votes for the politician only if their belief that the politician cares about the same issue as theirs is sufficiently high. This specialisation allows for the possibility that groups consist of receivers that care about different sets of states/issues.

I first give examples demonstrating that ensuring diversity within a group—both in terms of the member receivers' preferred states as well as their thresholds—can help the sender gain credibility in her communication. Semi-public communication benefits the sender by allowing her to take advantage of differently diverse groups from which she can gain credibility. However, I also show that splitting the audience into groups is potentially costly for the sender because she must give up hope of persuading at least one receiver in each group to gain credibility (Proposition 1).

I then show that it is possible for the sender to focus on trying to persuade receivers who are easier to persuade (Proposition 2), and that the sender's problem can be written as a multi-dimensional knapsack problem. To obtain further results, I consider the case when receivers are contentious; i.e., when no one argument can be persuasive to receivers who prefer different states. With such receivers, I show that the sender need only consider groups that contain at most one receiver who prefers any particular state, and that optimal communication within each group involves the sender expressing that they prefer a particular state (Proposition 3). In particular, the result implies that ambiguous communication that does not make clear the state is not credible. I also provide a simple algorithm that can attain an optimal partition when there are only two or three states (Proposition 4).

With single-minded receivers, private communication is never strictly preferred by the sender (Proposition 4). However, it is possible for the sender to strictly prefer public communication over any (other) semi-public communication. In fact, public communication is

[4] I discuss how single-minded receivers can be used as building blocks to model other receiver preferences in section 4.

the only way for the sender to nontrivially persuade the maximal number of receivers in any group, and the sender can achieve this only if the size of the audience is smaller than the number of states, each receiver prefers a different state, and receivers are not contentious (Proposition 5). I also show that public communication is guaranteed to be optimal as receivers become increasingly contentious (Proposition 6).

Although I mainly interpret my results in the context of a politician trying to persuade voters, my results are also applicable to other contexts. For example, a seller attempting to persuade buyers to purchase a product by sending individual or group emails, or a manager attempting to induce effort from her workers by holding a single meeting or several meetings. Finally, I emphasise that the distinguishing feature of my model is that the sender here is able to communicate in groups. While the literature has compared public versus private communication as well as a combination of private and public communication (e.g., Goltsman and Pavlov, 2011; Arieli and Babichenko, 2019; Mathevet, Perego and Taneva, 2020), the idea that the sender communicates in strategically formed groups is new to the literature.[5]

Related literature The rich literature on cheap talk began with Crawford and Sobel (1982) who consider the case with a single informed sender and a single receiver.[6] Farrell and Gibbons (1989) analyse a cheap-talk model with two receivers and shows, *inter alia*, that the sender can grain credibility by communicating publicly (instead of privately) due to a *mutual discipline* effect whereby the presence of one receiver disciplines the communication with the other receiver and vice versa (as in the example in the introductory paragraph). The results in this paper extend the idea of mutual discipline to a setting with more than two receivers, which allows for semi-public communications and richer sources of the disciplining effect.[7]

In Crawford and Sobel's (1982) and many other cheap-talk games, the sender's preference depends on the state so that the credibility of the sender's communication can arise from the endogenous cost of messages as in signalling games (Spence, 1973). In the paper, I focus instead on the case in which the sender's preference is state independent thus re-

[5] I briefly comment on the sender's ability to communicate via multiple partitions of the receivers in the discussion. Such an extension would include a combination of private and public communication as a special case.

[6] See surveys by Sobel (2013); Özdogan (2016); Kamenica (2019); Bergemann and Morris (2019); Forges (2020).

[7] Goltsman and Pavlov (2011) study the two-receiver version of Crawford and Sobel (1982)'s uniform-quadratic model. Battaglini (2002) studies cheap-talk models with multiple senders.

moving the possibility of signalling to generate credibility for the sender.[8] This allows me to focus on semi-public communication as the sole way in which the sender can gain credibility in her communication. A number of authors have shown alternative ways in which the sender with state-independent preferences in cheap talk models can gain credibility. Chakraborty and Harbaugh (2010) show that, when the state is multidimensional, a sender who faces a single receiver can gain credibility by trading off different dimensions of the state. Lipnowski and Ravid (2020) observe that a sender facing a single receiver gains credibility by degrading self-serving information. Schnakenberg (2015; 2017) studies how a sender facing multiple receivers can credibly communicate information to multiple receivers in collective choice settings (e.g., voting) by public cheap talk. Salcedo (2019) considers a similar problem in which a sender, who faces many receivers, cares about persuading only a subset of the receivers.

CSG problems have been studied extensively in the computer science literature (see, for example, a survey by Rahwan et al., 2015). In economics, Sandholm (1999) shows that the determination of winners in combinatorial auctions is a CSG problem. Although general CSG problems have been shown to be computationally hard to solve, the literature has identified classes of CSG problems that are tractable. Of particular relevance is a class of CSG problems that have *agent-type representation* (Aziz and Keijzer, 2011; Ueda et al., 2011) meaning that the value of coalition of players depend on the types of players that the coalition contains. I describe how the Sender's optimal communication can be computed using algorithms from the CSG literature.

1 An illustrative example

To develop some intuition for the results, suppose that the sender is a politician who faces $n \in \mathbb{N}$ voters (i.e., receivers), $N = \{1, 2, \ldots, n\}$. The politician and the voters care about one of three possible issues (i.e., states of the world), $\Theta = \{\theta^1, \theta^2, \theta^3\}$. Let μ_0 denote the common prior belief about the politician's *opinion* defined as the issue $\theta \in \Theta$ that the politician cares about. Suppose that each receiver $i \in N$ is a single-issue voter and votes for the politician only if he believes that the likelihood that the politician shares his opinion is sufficiently high. Specifically, let $\theta_i \in \Theta$ denote voter $i \in N$'s opinion and suppose the

[8] When the sender's preference is state independent, the receivers have complete information about the sender's preference. Hence, some authors describe such a sender as having an extreme bias (Chakraborty and Harbaugh, 2010) or transparent motives (Lipnowski and Ravid, 2020).

receiver votes for the politician only if his belief that $\theta = \theta_i$ is greater than a common threshold $\gamma = \frac{3}{4}$. Initially, no receiver would vote for the politician (i.e., $\mu_0(\theta) < \gamma$ for all $\theta \in \Theta$) and the politician's objective is to maximise the number of votes using cheap-talk communication; i.e., by sending messages that are costless and are correlated with her opinion.

The first example demonstrates how the sender's communication can be credible when there is a diversity of opinion in the audience.

Example 1. Suppose that there are three voters, $\theta_i = \theta^i$ for all $i \in N$, and $\mu_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; i.e., each voter $i \in \{1,2,3\}$'s opinion is given by θ^i and they believe that the politician's opinion is equally likely to be any one of the three possible issues. Standard arguments mean that the politician cannot persuade any voter to vote for her in equilibrium using private communication.[9] However, the politician can guarantee herself one vote in equilibrium by publicly communicating her opinion truthfully. This is because truthful public communication results in her obtaining exactly one vote independently of whether she tells the truth or lies about her opinion. Consequently, the politician has no incentive to lie about the state. Moreover, in this example, no other communication (in groups or otherwise) can guarantee a strictly higher number of votes for the politician; i.e., telling the public her true opinion is, in fact, optimal for the politician. □

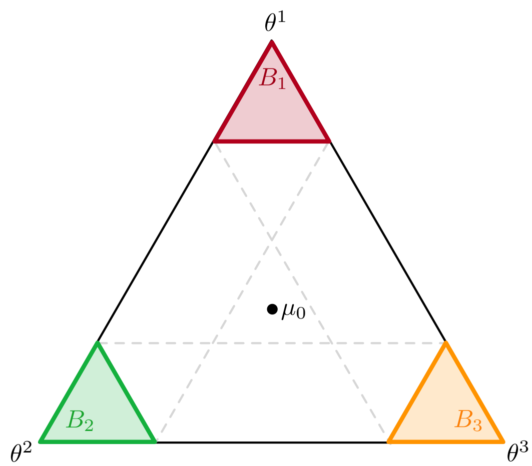
In Example 1, public communication is credible because the diversity of opinion among the voters disciplines the politician's communication. For example, when the politician's opinion is θ^1, her incentive to lie about her opinion to either voter 2 or 3 (whose opinions are θ^2 and θ^3, respectively) is offset by her incentive to be truthful to voter 1.[10] To see this in another way, let us take a belief-based approach and express the politician's communication by the distribution of posterior beliefs that it induces if it were credible. Note that any belief about the politician's opinion can be represented as a point in the belief simplex as shown in Figure 1, where each vertex labelled $\theta \in \Theta$ corresponds to the receiver(s) having a certain belief that θ is the politician's opinion. The prior belief, μ_0, in Example 1 lies in the centre of the simplex. For each $i \in N$, the shaded region labelled

[9]To see this, by way of contradiction, suppose there was a message sent privately to each voter that can persuade a receiver to vote in an equilibrium. The politician then has the incentive to always send this persuasive message independently of her true opinion. But such a message is necessarily uninformative about the politician's opinion and, given the assumption on the prior belief, the receiver would not be persuaded—contradicting the initial assertion that the message was persuasive.

[10]Note that the politician can obtain at most one vote by lying.

$B_i \coloneqq \{\mu \in \Delta\Theta : \mu(\theta_i) \geq \frac{3}{4}\}$ represents the voter i's *voting region*; i.e., the set of beliefs under which voter i would vote for the politician.[11]

Figure 1: Example 1.



Recall that a set of posterior beliefs can be induced by some credible communication if and only if the convex hull of the set contains the prior belief μ_0 (Aumann and Maschler, 1968; Kamenica and Gentzkow, 2011). This condition is also necessary for the politician's cheap talk to be able to induce a set of posterior beliefs. However, to ensure that cheap talk is credible, it must also be the case that the politician is indifferent among all beliefs in the set of posterior beliefs that her communication would induce if it were credible. Combining these observations gives that the politician is able to always obtain one vote in this Example 1 if the convex hull of the voting regions contains the prior (Schnakenberg, 2015); i.e., $\mu_0 \in \text{co}(B_1 \cup B_2 \cup B_3)$, where $\text{co}(\cdot)$ denotes the convex hull of a set.[12] The condition also implies that the politician must express a particular stance in order to be credible—ambiguous communication that does not make clear the politician's opinion is not credible.

Figure 1 also makes it clear that that the politician is unable to obtain votes from any pairs of voters because $\mu_0 \notin \text{co}(B_i \cup B_j)$ for any distinct $i, j \in \{1, 2, 3\}$. This, in particular, implies that only the maximally diverse group of voters can give rise to the disciplining effect that gives credibility to the politician's communication. Hence, in Example 1, com-

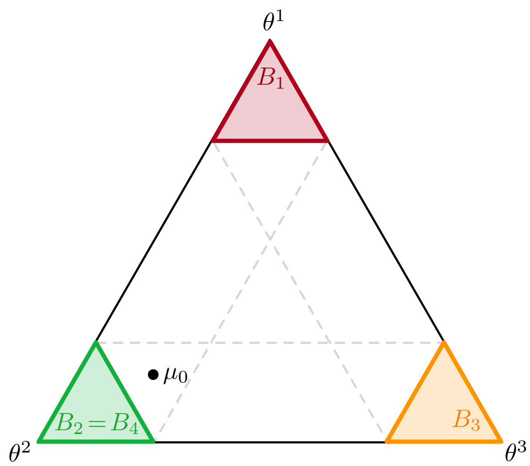
[11] Since $\mu_0(\theta) < \frac{3}{4}$ for any $\theta \in \Theta$, the prior belief μ_0 is not contained in any of the voting regions. Moreover, the fact that the threshold exceeds $\frac{1}{2}$ means that the voting regions do not intersect; i.e., $B_i \cap B_j = \varnothing$ for all distinct $i, j \in N$.

[12] That the voting regions do not intersect also means that no posterior belief can induce more than one vote. Hence, the politician cannot do strictly better than truthful public communication that always induces one vote.

municating in groups (e.g., talking to voters 1 and 2 together while talking to voter 3 separately) is unhelpful for the politician. The next example demonstrates that, when differently diverse subsets of the audience can give rise to the disciplining effect, the sender can benefit from communicating in groups because doing so allows her to tailor her communication to each group.

Example 2. Suppose now that the politician faces an additional voter who has the same opinion as voter 2 (i.e., $\theta_2 = \theta_4 = \theta^2$), and that the prior belief is given by $\mu_0 = (\frac{1}{5}, \frac{3}{5}, \frac{1}{5})$. Figure 2 shows the voting regions as well as the prior belief in the belief simplex.

Figure 2: Example 2.



Notice that, because $\mu_0 \in \text{co}(B_1 \cup B_2)$, the politician is able to always obtain one vote by communicating with voters 1 and 2 together. Similarly, because $\mu_0 \in \text{co}(B_3 \cup B_4)$, the politician is able to always obtain one vote by communicating with voters 3 and 4 together. In other words, by partitioning the audience into two groups, $\{\{1,2\},\{3,4\}\}$, she is able to always obtain two votes.[13] In contrast, public communication can only guarantee at most one vote and the politician cannot obtain any vote with private communication.[14] It follows that the politician's optimal communication involves splitting the audience into two groups and communicating publicly within each group but privately across the two groups. $\square$

In Example 2, unlike in the first example, preferences and the prior belief are such that the politician is able to guarantee a vote from three differently diverse groups of voters: by

[13] To be concrete, letting σ_{ij} denote the politician's communication strategy with respect to the pair $\{i, j\}$ of voters and $M := \{m_0, m_1\}$, an optimal communication strategy profile is: $\sigma_{12}(m_0|\theta^1) = \sigma_{34}(m_0|\theta^3) = 0$, $\sigma_{12}(m_0|\theta^2) = \sigma_{34}(m_0|\theta^2) = 1$ and $\sigma_{12}(m_0|\theta^3) = \sigma_{34}(m_0|\theta^1) = \frac{3}{4}$.

[14] To ensure that public communication is credible, the politician must ensure that voters 2 and 4 are indifferent between voting or not voting (and be able to choose an equilibrium such that the expected number of votes she gets is one). In particular, this means that truthful communication is no longer credible.

pairing voters whose opinion is θ^2 with those whose opinion is θ^1 or θ^3, or by a group that contains receivers of all possible opinions. Moreover, any communication that guarantees a vote from the pair consisting of voters 1 and 2 would not guarantee a vote from the pair consisting of voters 3 and 4; because whatever would induce voter 1 to vote would not induce voter 4 to vote. Communicating in groups is thus strictly preferred over public communication because it allows the politician to simultaneously gain credibility by having sufficient diversity in each group and to tailor her communication to what each group finds persuasive.

A useful observation from the examples above is that the politician's communication across groups can be assumed to be independent (conditional on the politician's opinion). Consequently, an optimal partition of voters can be obtained in two steps. First, one can associate a value, $w(G)$, to any group $G \subseteq N$ of voters based on the number of votes that the politician can guarantee from the group using the aforementioned geometric condition. The problem of finding an optimal communication reduces to finding a partition of voters that maximises the sum of values of groups, which is a type of CSG problem. In what follows, I set up a model of cheap-talk persuasion game with multiple receivers that allows for the sender's problem to be reduced to a CSG problem, and derive properties of optimal partitions under different specifications of receivers' preferences.

2 A cheap-talk game with multiple receivers

There is a single *Sender*, denoted S, and a finite set $N := \{1,2,\ldots,n\}$ of *Receivers*. Each Receiver $i \in N$ chooses an action a_i from a finite set A_i, and his payoff depends only on his own action and the state of the world $\theta \in \Theta$, where Θ is a finite set of states. The Sender's payoff is state independent, separable with respect to each Receiver's action, and Receivers' actions are weakly beneficial for the Sender.[15] Thus, each Receiver $i \in N$'s payoff is given by $u_i : A_i \times \Theta \to \mathbb{R}$ and the Sender's payoff is given by $u_S : A \to \mathbb{R}_+$ with $u_S(a_1,\ldots,a_n) := \sum_{i\in N} v_i(a_i)$, where $A := \times_{i\in N} A_i$ and $v_i : A_i \to \mathbb{R}_+$ for all $i \in N$. The Sender has a strict preference towards each Receiver taking a particular action; i.e., $\overline{v}_i^* := \max_{a_i \in A_i} v_i(a_i) > 0$ for all $i \in N$. The majority of the results concern the case in which each Receiver's action is binary and the Sender wishes to maximise the number of Receivers taking the higher

[15] I discuss how to accommodate cases in which the Sender's payoff is not separable in Receivers' actions in section 4.

action.; i.e., $A_i := \{0,1\}$ and $v_i(a_i) := a_i$ for all $i \in N$.[16] Throughout, I refer to any nonempty subset $G \subseteq N$ of Receivers as a *group* of Receivers.

The timing of the game is as follows. First, the Sender publicly chooses a partition $\mathscr{P} \in \Pi(N)$ of Receivers.[17] Then, the Sender observes the state drawn according to a common full-support prior distribution $\mu_0 \in \Delta\Theta$, and sends a message $m_G \in M$ to each group $G \in \mathscr{P}$ of Receivers, where M is a set of possible messages that is sufficiently rich.[18] Each Receiver $i \in N$ who belongs in group $G_i \in \mathscr{P}$ (only) observes the Sender's message to group G_i, m_{G_i}, and then takes an action $a_i \in A_i$. Payoffs are then realised.

Given any $\mathscr{P} \in \Pi(N)$, I define a *$\mathscr{P}$-equilibrium* as a weak perfect Bayesian equilibrium of the game in which the Sender is restricted to sending the same message to Receivers who belong in the same group.[19] Formally, a $\mathscr{P}$-equilibrium is a collection of three maps $(\sigma^{\mathscr{P}}, \alpha, \mu^{\mathscr{P}})$, where $\sigma^{\mathscr{P}} : \Theta \to \Delta(M^{\mathscr{P}})$ denotes the Sender's messaging strategy, $\alpha = (\alpha_i)_{i \in N}$ denotes the Receivers' action strategy profile with $\alpha_i : M \to \Delta A_i$ for each $i \in N$, and $\mu^{\mathscr{P}} = (\mu_G)_{G \in \mathscr{P}}$ denotes the belief map profile for Receivers with $\mu_G^{\mathscr{P}} : M \to \Delta\Theta$ for each group $G \in \mathscr{P}$. For brevity, I write $\sigma \equiv \sigma^{\mathscr{P}}$ and $\mu \equiv \mu^{\mathscr{P}}$ when no confusion should arise, and call a tuple (σ, α, μ) a *$\mathscr{P}$-equilibrium* if it satisfies the following conditions: (i) for each $G \in \mathscr{P}$, the belief map μ_G is derived by updating μ_0 via Bayes rule whenever possible, i.e, for all $m_G \in M$,

$$\mu_G(\cdot|m_G) \sum_{\theta \in \Theta} \sum_{m_{-G} \in M_{-G}} \sigma(m_G; m_{-G}|\theta)\, \mu_0(\theta) = \sum_{m_{-G} \in M_{-G}} \sigma(m_G; m_{-G}|\cdot)\, \mu_0(\cdot),$$

where $M_{-G} := M^{\mathscr{P} \setminus \{G\}}$; (ii) each receiver $i \in N$'s action strategy α_i is optimal given μ_{G_i}, i.e., for all $i \in N$ and all $m_{G_i} \in M$,

$$\text{supp}\left(\alpha_i(\cdot|m_{G_i})\right) \subseteq \arg\max_{a_i \in A_i} \sum_{\theta \in \Theta} u_i(a_i, \theta)\, \mu_{G_i}(\theta|m_{G_i}); \tag{1}$$

[16] Given a set X, let $\Pi(X)$ denote the set of all partitions of X. Given a set X, ΔX denotes the set of probability measures over the set X. Given a probability measure $\mu \in \Delta X$, $\text{supp}(\mu)$ denote the support of measure μ.

[17] See section 4 for the case in which the Sender chooses a partition after observing the state, as well as the case in which each Receiver knows the members of the group that he belongs in but does not know how Receivers outside of his group are partitioned.

[18] It will suffice that $|M| \geq |\Theta|$.

[19] One can interpret the Sender's ability to communicate in groups as a limited form of commitment to communication strategies. In particular, given a partition, the sender is able to commit to communication strategies that send the same message to Receivers who are in the same group.

(iii) Sender's messaging strategy σ is incentive compatible given α, i.e., for all $\theta \in \Theta$,

$$\operatorname{supp}(\sigma(\cdot|\theta)) \in \arg\max_{m \in M^{\mathscr{P}}} \sum_{a \in A} u_S(a) \prod_{G \in \mathscr{P}} \prod_{i \in G} \alpha_i(a_i|m_G). \tag{2}$$

I refer to a pair $(\mathscr{P}, \sigma^{\mathscr{P}})$ as the Sender's *communication strategy*. I say that the Sender's communication strategy is *public* if $\mathscr{P} = \{N\}$, *private* if $\mathscr{P} = \{\{i\}\}_{i \in N}$, and *strictly semi-public* if it is neither private nor public.

Let $W^*_{\mathscr{P}}$ denote the Sender's payoff in her preferred $\mathscr{P}$-equilibrium, and w^*_G denote the Sender's payoff in a Sender-preferred equilibrium of the public cheap-talk game with a group $G \subseteq N$ of Receivers. The following lemma establishes that I can compute $W^*_{\mathscr{P}}$ by solving the Sender's the public cheap-talk problem with respect to each group $G \in \mathscr{P}$ independently.[20] The result follows from the fact that, given the separability of the Receivers' actions in the Sender's payoff, the Sender does not benefit from correlating messages across groups.

Lemma 1. *Given a partition $\mathscr{P} \in \Pi(N)$, the Sender's payoff in her preferred $\mathscr{P}$-equilibrium is given by the sum of her payoffs in her preferred equilibrium of the public cheap-talk game with each group $G \in \mathscr{P}$ of Receivers; i.e., $W^*_{\mathscr{P}} = \sum_{G \in \mathscr{P}} w^*_G$.*

Proof. See Appendix A.1. ■

The lemma allows me to write the Sender's optimal communication strategy with respect to any group $G \subseteq N$ of Receivers as a solution to the following problem:

$$W^*(G) := \max_{\mathscr{P} \in \Pi(G)} \sum_{F \in \mathscr{P}} w(F), \tag{3}$$

where $w : 2^N \to \mathbb{R}$ by $w(G) := w^*_G$ for any group $G \subseteq N$ of Receivers and $w(\varnothing) := 0$.

To proceed, I first characterise the coalition function $w(\cdot)$ by adopting a belief-based approach (Aumann and Maschler, 1968; Kamenica and Gentzkow, 2011). To that end, let $V_G : \Delta\Theta \rightrightarrows \mathbb{R}$ be a correspondence such that $V_G(\mu)$ gives the set of payoffs the Sender can attain from a group $G \subseteq N$ of Receivers who best responds given a common belief $\mu \in \Delta\Theta$ under some tie-breaking rules for the Receivers.[21] I refer to the correspondence V_G as the

[20] Arieli and Babichenko (2019) obtains an analogous (and stronger) result (Theorem 4) in the case of Bayesian persuasion (i.e., without requiring incentive compatibility condition for the Sender, (2), in the definition of $\mathscr{P}$-equilibrium) with binary states and binary actions.

[21] More concretely, the correspondence V_G is given by $V_G(\cdot) := \sum_{i \in G} V_i(\cdot)$, where $V_i : \Delta\Theta \rightrightarrows \mathbb{R}$ is defined

Sender's *value correspondence with respect to group G of Receivers*. Given any group $G \subseteq N$ of Receivers, define $B_G : \mathbb{R} \rightrightarrows \Delta\Theta$ via $s \mapsto \{\mu \in \Delta\Theta : \max V_G(\mu) \geq s\}$, and say that a payoff $s \in \mathbb{R}$ is G-securable if $\mu_0 \in \overline{\text{co}}(B_G(s))$. Observe that any group $G \subseteq N$ of Receivers can be thought of as a single representative Receiver whose preference is such that the Sender's value correspondence with respect to the representative Receiver is given by V_G. Therefore, Lipnowski and Ravid's (2020) Corollary 1 implies that the Sender can attain a payoff $s \geq \max V_G(\mu_0)$ using public cheap talk among group $G \subseteq N$ of Receivers if and only if s is G-securable, and that $w(G)$ is given by the highest payoff that is G-securable for any group $G \subseteq N$. The lemma below lists some useful properties of w. The first property gives a bound on the Sender's payoff from any group, the second tells us that the coalition function is increasing (in the set inclusion order). The last property, in particular, implies that adding a Receiver $i \in N$ to a group cannot increase the Sender's payoff more than the highest payoff that the Sender can attain from i, $\overline{v}_i^*$.

Lemma 2. *The coalition function $w(\cdot)$ has the following properties.*

(i) $\max V_G(\mu_0) \leq w(G) \leq \overline{v}_G^* := \sum_{i \in G} \overline{v}_i^*$.

(ii) $w(G) \geq w(G')$ *for all* $G' \subseteq G \subseteq N$.

(iii) $w(G) \leq w(G') + \sum_{i \in G \setminus G'} \overline{v}_i^*$ *for all* $G' \subseteq G \subseteq N$.

Proof. Property (i) follows from the fact that $\max V_G(\mu_0)$ is the Sender's payoff in her preferred babbling equilibrium of a public cheap talk game with group G of Receivers and that $\max V_i \leq \overline{v}_i^*$ for any $i \in N$. (ii) follows from the fact that $v_i(\cdot) \geq 0$ for all $i \in N$ so that, for any $G' \subseteq G \subseteq N$, $\max V_G = \max V_{G'} + \max V_{G \setminus G'} \geq \max V_{G'}$ implying $B_{G'}(s) \subseteq B_G(s)$. (iii) holds because $\max V_G + \sum_{i \in G' \setminus G} \overline{v}_i^* \geq \max V_G + \max V_{G' \setminus G} = \max V_{G'}$ for any $G' \subseteq G \subseteq N$. ■

Let $N_0 \subseteq N$ denote the set of all Receivers who are willing to take the sender-preferred action without any information (i.e., $i \in N_0$ if and only if $\max V_i(\mu_0) = \overline{v}_i^*$). Thus, N_0 is the group of Receivers consisting of all those that do not need persuading. Let $N_1 := N \setminus N_0$ denote the group of Receivers consisting of all those that do need persuading. The

as

$$\mu \mapsto \text{co}\left(\left\{ v_i(a_i) : a_i \in \underset{a_i' \in A_i}{\arg\max} \int_\Theta u_i(a_i', \theta)\, \mathrm{d}\mu(\theta) \right\}\right).$$

Since V_i is a Kakutani correspondence (by Berge's theorem), V_G is also a Kakutani correspondence. In particular, $\max V_G$ is a well-defined, upper semi-continuous function.

following result establishes that the Sender can treat the problem of persuading Receivers in N_1 independently of persuading Receivers in N_0. In other words, there is never any need for the Sender to "preach to the choir."

Theorem 1 (Don't Preach to the Choir)**.** *The Sender's problem can be split into two independent problems of communicating with Receivers who do not require persuading (i.e., set N_0 of Receivers) and those who require persuading (i.e., set N_1 of Receivers); i.e.,*

$$W^*(N) = W^*(N_0) + W^*(N_1)$$

Moreover, $W^*(N_0) = \overline{v}^*_{N_0}$ and the Sender can attain this payoff with any partition of N_0.

Proof. The second part of the theorem is immediate: Since N_0 consists of Receivers whose optimal action under the prior belief includes the Sender-preferred action, a payoff of $\overline{v}^*_G$ is G-securable with any group $G \subseteq N_0$. To prove the first part of the theorem, let $\mathscr{P}_1^* \in \Pi(N_1)$ be a maximiser that attains $W^*(N_1)$ and $\mathscr{P}^* \in \Pi(N)$ be a maximiser that attains $W^*(N)$. Since $\{N_0, N_1\} \in \Pi(N)$, by definition, $W^*(N) \geq W^*(N_0) + W^*(N_1)$. Now take any $G \in \mathscr{P}^*$ such that $G \cap N_0 \neq \varnothing$. Then, part (iii) of Lemma 2, $w(G) - w(G \backslash (G \cap N_0)) \leq \sum_{i \in G \cap N_0} \overline{v}^*_i$. Since this holds for all $G \in \mathscr{P}^*$,

$$\begin{aligned} W^* = \sum_{G \in \mathscr{P}^*} w(G) &\leq \sum_{G \in \mathscr{P}^*} w(G \backslash (G \cap N_0)) + \sum_{G \in \mathscr{P}^*} \sum_{i \in G \cap N_0} \overline{v}^*_i \\ &= \sum_{G \in \mathscr{P}^*} w(G \backslash (G \cap N_0)) + \sum_{i \in N_0} \overline{v}^*_i \\ &\leq W^*(N_1) + W^*(N_0), \end{aligned}$$

where the last inequality follows from the fact that $\{G \backslash (G \cap N_0)\}_{G \in \mathscr{P}^*}$ is a partition of N_1. ■

The theorem means that the Sender can never strictly benefit by grouping together Receivers from N_0 and N_1, and that the Sender's problem is nontrivial only with respect to the set N_1 of Receivers. The following result, which is immediate from part (ii) of Lemma 2, further implies that the Sender's problem with respect to the set N_1 of Receivers is nontrivial only if publicly communicating with N_1 is beneficial for the Receiver.

Corollary 1. *If $w(N_1) = 0$, then $W^*(N_1) = 0$; i.e. if public communication is not beneficial for the Sender with respect to those who require persuading, then she cannot benefit from semi-public communication either.*

Proof. Since $v_i(\cdot) \geq 0$, $w(G) \geq 0$ for all $G \subseteq N_1$. If $0 = w(N_1)$, part (ii) of Lemma 2 implies that $0 = w(N_1) \geq w(G)$ for all $G \subseteq N_1$. Hence, $w(G) = 0$ for all $G \subseteq N_1$ so that $W^*(N_1) = 0$. ■

Note that the pair $(N, w(\cdot))$ defines a coalition game with transferable utility (a type of a cooperative game), where $w(\cdot)$ is the coalition function that gives the value of any sub-coalition (i.e., a group of Receivers).[22] Viewed in this way, the Sender's problem, (3), is an example of *coalition structure generation (CSG) problems* that seek to find a coalition structure (i.e., a partition of Receivers) that maximises the total value of sub-coalitions.[23] While in principle, CSG problems can be solved by evaluating every possible partition of players, such a brute-force approach is not practicable as the number of possible partitions—i.e., the cardinality of the set $\Pi(N)$—grows (double) exponentially with the number of players.[24] The solution to a CSG problem is immediate if the coalition function is supper-additive or sub-additive: in the former case, the optimal partition is the "grand coalition" (i.e., $\mathscr{P} = \{N\}$), while in the latter case, the optimal partition is the trivial coalitions (i.e., $\mathscr{P} = \{\{i\}\}_{i \in N}$). However, as examples in section 1 demonstrate, the coalition function $w(\cdot)$ is neither super-additive nor sub-additive.[25] In fact, finding a solution to CSG problems in general is NP-hard (Sandholm et al., 1999). Thus, to obtain further properties of optimal communication, I now consider Receivers who are "single-minded" as in the introductory example.

3 Single-minded Receivers

Suppose now that Receivers' actions are binary, $a_i \in A_i := \{0, 1\}$ for all $i \in N$, and that Receivers are *single-minded*; i.e., each Receiver $i \in N$ takes action $a_i = 1$ (resp. $a_i = 0$) if he believes that the state is $\theta_i \in \Theta$ is greater (resp. smaller) than probability $\gamma_i \in [0, 1]$.[26] Thus, each Receiver $i \in N$'s preferences can be described by a pair of parameters $(\theta_i, \gamma_i) \in$

[22] For more on coalition games, see, for example, Peleg and Sudhölter (2007). To be clear, the game that I study is not a cooperative game since it is the Sender who receives the value of the coalition as opposed to the Receivers who are the members of coalitions.

[23] CSG problems have been extensively studied by computer scientists in the context of multi-agent systems. See Rahwan et al. (2015) for a survey.

[24] See Proposition 1 in Sandholm et al. (1999). For example, even with $n = 15$ players, the number of possible partitions exceeds 1 billion.

[25] In section 4, I discuss how existing algorithms can be used to solve the Sender's problem.

[26] For example, $u_i(a_i, \theta) := a_i(\mathbb{1}_{\{\theta = \theta_i\}} - \gamma_i)$, where the Receiver's payoff from choosing $a_i = 0$ is normalised to be zero.

$\Theta \times (0,1)$, where θ_i is the Receiver i's *opinion* (i.e., his preferred state of the world) and γ_i denotes Receiver i's strength of his opinion. Let $B_i := \{\mu \in \Delta\Theta : \mu(\theta_i) \geq \gamma_i\}$ denotes the set of beliefs under which it is optimal for the Receiver $i \in N$ to takes action optimally, and $B_G := \bigcap_{i \in F} B_i$ for any group $G \subseteq N$. Given two distinct Receivers $i, j \in N$ who share an opinion (i.e., $\theta_i = \theta_j$), I say that Receiver i is *more* (resp. *less*) *extreme* than Receiver j if $\gamma_i \geq \gamma_j$ (resp. $\gamma_i \leq \gamma_j$).

Lemma 3. *With single-minded Receivers, a payoff $s \in \mathbb{R}$ is G-securable if and only if*

$$\mu_0 \in \overline{\text{co}} \left(\bigcup_{F \subseteq G : |F| = \lceil s \rceil} B_F \right). \tag{4}$$

where $\lceil s \rceil$ denotes the smallest integer that is greater than s. Moreover, μ_0 is at most a convex combination of $|\Theta|$-many elements of $\{B_F\}_{F \subseteq G : |F| = \lceil s \rceil}$.

Proof. For any group $G \subseteq N$ of single-minded Receivers,

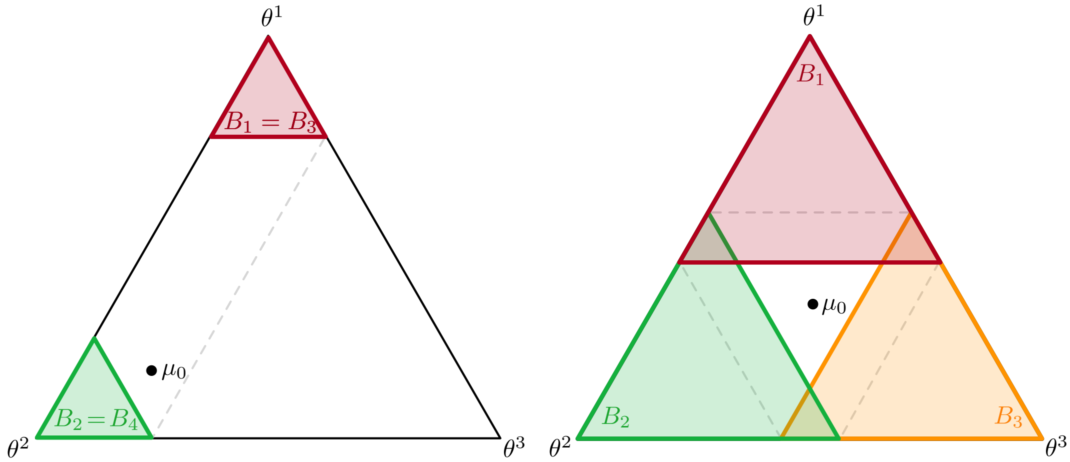
$$\begin{aligned} B_G(s) &= \left\{ \mu \in \Delta\Theta : \sum_{i \in G} \mathbf{1}_{\{\mu(\theta_i) \geq \gamma_i\}} \geq s \right\} = \left\{ \mu \in \Delta\Theta : \sum_{i \in G} \mathbf{1}_{\{\mu(\theta_i) \geq \gamma_i\}} \geq \lceil s \rceil \right\} \\ &= \{\mu \in \Delta\Theta : \exists F \subseteq G,\ \mu \in B_F \text{ and } |F| = \lceil s \rceil\} = \bigcup_{F \subseteq G : |F| = \lceil s \rceil} B_F \end{aligned}$$

so that (4) follows. The last result follows from Carathéodory's theorem while noting that $\Delta\Theta$ has dimension $|\Theta| - 1$. ■

When concerning whether a payoff of $s = 1$ is G-securable, the condition (4) reduce to $\mu_0 \in \overline{\text{co}}(\bigcup_{i \in G} B_i)$, which was the relevant condition for Examples 1 and 2. Following the setup of these examples, Figure 3 below gives two cases in which the Sender is able to attain a payoff of 2 from a group G of workers. In panel (a), the voting regions of Receivers with different opinions do not overlap, and the Sender is able always to obtain two votes because $\mu_0 \in \overline{\text{co}}(B_{\{1,3\}} \cup B_{\{2,4\}})$. In panel (b), the voting regions of Receivers with different opinions are overlapping, and the Sender is able to always obtain two votes because $\mu_0 \in \overline{\text{co}}(B_{\{1,2\}} \cup B_{\{2,3\}} \cup B_{\{1,3\}})$. Note that the Sender can use public communication to attain a payoff of two in both cases.

Figure 3: $w(G) = 2$.

(a) Non-overlapping voting regions. (b) Overlapping voting regions.



The following lemma summarises some additional properties of the coalition function when Receivers are single-minded. The first property implies that the Sender's payoff from a group is simply the number of Receivers that the Sender can persuade to take action. The second property is that diversity of opinions in an audience is necessary for the Sender's communication to be persuasive. The third property says that the Sender can persuade all Receivers in a group to take action if and only if the group consists of Receivers who do not require persuading. The fourth property concerns groups consisting of Receivers who require persuading and gives the upper bound on the proportion of Receivers that the Sender can possibly persuade. This property gives an upper bound on the size of the group that depends on the payoff that the Sender can attain from a group of Receivers.[27] The final property is that replacing a Receiver in a group with another less extreme Receiver with the same opinion does not affect the Sender's payoff.

Lemma 4. *Suppose Receivers are single minded. The coalition function $w(\cdot)$ has the following properties.*

(i) $w(G) \in \{0, 1, \ldots, |G|\}$ *for all* $G \subseteq N$.

(ii) $w(G) = 0$ *for any* $G \subseteq N_1$ *such that* $\theta_i = \theta_j$ *for all* $i, j \in G$.

[27] In a working version of this paper, I show that the upper bounds in parts (iv) and (v) in the lemma are tight; i.e., there exists $\mu_0 \in \Delta\Theta$ and a set of single-minded Receivers that require persuading that attains these upper bounds.

(iii) $w(G) = |G|$ *if and only if* $\mu_0 \in B_i$ *for all* $i \in G$ *(i.e.,* $G \subseteq N_0$*).*

(iv) For any $G \subseteq N_1$,

$$w(G) \leq \begin{cases} |G|-1 & \text{if } |G| \leq |\Theta| \\ \left\lceil |G| \frac{|\Theta|-1}{|\Theta|} - 1 \right\rceil \leq |G|-1 & \text{if } |G| > |\Theta| \end{cases}. \tag{5}$$

(v) For any non-singleton group $G \subseteq N_1$ *and any* $k \in \{1, \dots, |G|-1\}$, *if* $w(G) = |G| - k$, *then* $|G| \leq k|\Theta|$.

(vi) $w(G) = w((G \backslash \{i\}) \cup \{j\})$ *if* j *is such that* $\theta_j = \theta_i$, $\gamma_i \geq \gamma_j$ *and* $\mu_0 \notin B_j$.

Proof. The first part follows from the fact that $\max V_G(\mu) = \sum_{i \in G} \mathbf{1}_{\{\mu(\theta_i) \geq \gamma_i\}}$ and takes positive integer values. That $\mu_0 \in B_i$for all $i \in G$ implies $w(G) = |G|$ is clear from (4). To see the converse, suppose that $w(G) = |G|$, then (4) implies $\mu_0 \in \overline{\text{co}}(\bigcap_{i \in G} B_i) = \bigcap_{i \in G} B_i$, where the equality follows from the fact that B_i is convex. Hence, $\mu_0 \in B_i$for all $i \in G$. (iii) Suppose that a nonempty $G \subseteq N_1$ consists only of Receivers with opinion $\theta \in \Theta$. Let $i^* \in G$ denote (any one of) the least extreme Receiver in G. Then, $B_i \subseteq B_{i^*}$ for all $i \in G$. Thus, if (4) holds for any $s \in \mathbb{R}_+$ such that $\lceil s \rceil \geq 1$, then $\mu_0 \in B_{i^*}$, which contradicts that $G \subseteq N_1$. I prove parts (iv) and (v) in Appendix A.1 by effectively converting the condition in Lemma 3 to bipartite graphs with specific properties. The last property follows from the fact that $B_i \subseteq B_j$. ■

Because a group consisting of a single Receiver is necessarily not diverse, part (ii) of Lemma 4 implies that the Sender cannot persuade a Receiver from N_1 to take action by communicating privately. In other words, Sender's payoff from privately communicating is the same as if she did not communicate at all.

Corollary 2. *Suppose Receivers are single-minded. Then, private communication is optimal if and only if there is no benefit from communication, i.e.,*

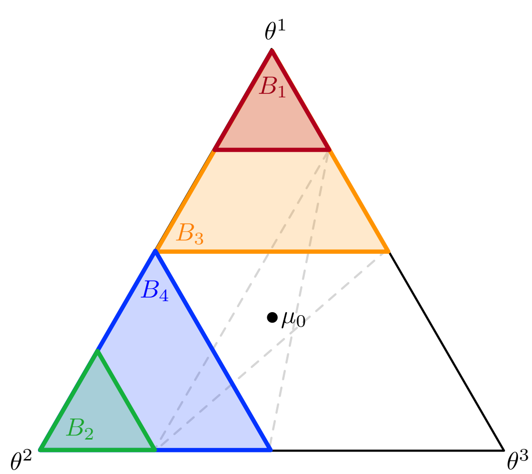
$$\mu_0 \notin \overline{\text{co}} \left(\bigcup_{i \in N_1} B_i \right). \tag{6}$$

Proof. Since private communication is always optimal among N_0, it suffices to characterise the condition when private communication is optimal among N_1. It only remains to show that the condition above holds if and only if there is no benefit from communication. First, note that if the condition above does not hold, then $w(N_1) \geq 1$ so that there is benefit from communication. Conversely, suppose there is some benefit from communication. Then, there exists a group $G \subseteq N_1$ such that $w(G) \geq 1$. By part (ii) of Lemma 2, $w(N_1) \geq w(G) \geq 1$. Note that any group that can G-secure a payoff $s > s'$ can also secure a payoff of s' so that a payoff of 1 is N_1-securable; i.e., $\mu_0 \in \overline{\text{co}}(\bigcup_{i \in N_1} B_i)$. ■

While private communication is never strictly preferred by the Sender, there are cases in which public communication is strictly preferred over strictly semi-public communication and vice versa. Indeed, public communication was strictly preferred for the Sender in Example 1 while semi-public communication was strictly preferred in Example 2. In the case of Example 2, strictly semi-public communication allowed the Sender to benefit from communicating with two groups that were differently diverse in terms of opinions. The next example demonstrates that diversity in terms of strength of opinions can also lead the Sender to strictly prefer strictly semi-public communication.

Example 3. As in the illustrative examples from section 1, there are three possible issues. There are four voters: (odd-numbered) voters 1 and 3 who share an opinion θ^1, and (even numbered) voters 2 and 4 who share an opinion θ^2. Suppose further that $\gamma_1 = \gamma_2 = \frac{3}{4}$ and $\gamma_3 = \gamma_4 = \frac{1}{2}$ so that voter 1 (resp. 2) is less extreme than voter 3 (resp. 4). Suppose that each voter believes that the politician's opinion is equally likely to be any one of the three possible issues. Figure 4 shows the voting regions as well as the prior belief in the belief simplex.

Figure 4: Example 3.



Observe that μ_0 is contained in $\overline{\text{co}}(B_1 \cup B_4)$ and $\overline{\text{co}}(B_2 \cup B_3)$ so that $w(\{1,4\}) = w(\{2,3\}) = 1$. Thus, the Sender can attain a payoff of 2 using strictly semi-public communication.

However, if the Sender were to use public communication, the Sender cannot attain a payoff of two because μ_0 is not contained in $\overline{\text{co}}(B_1 \cup B_2)$. Thus, although groups $\{1,4\}$ and $\{2,3\}$ are equally diverse in terms of opinions, the Sender can nevertheless strictly prefer strictly semi-public communication due to the diversity of strength of opinions across the groups. □

Examples 2 and 4 demonstrate together that diversity in the audience—both in terms of opinions and strength of opinions—can be the reason why the Sender prefers semi-public communication. Intuitively, with diversity in the audience, different arguments (i.e., messaging strategies) are persuasive to different subsets of the audience and semi-public communication allows the Sender to tailor her communication to subsets of the audience. However, splitting the audience into groups is potentially costly for the Sender as the following result shows.

Proposition 1. *The Sender's potential payoff from communicating semi-publicly with a collection of single-minded Receivers who require persuading is decreasing in the number of groups she forms. In fact,*

$$W^*_{\mathscr{P}_1} \leq |N_1| - |\mathscr{P}_1| \ \forall \mathscr{P}_1 \in \Pi(N_1)$$

Proof. From part (iv) of Lemma 4, implies that, for any $\mathscr{P}_1 \in \Pi(N_1)$,

$$W^*_{\mathscr{P}_1} = \sum_{G \in \mathscr{P}} w(G) \leq \sum_{G \in \mathscr{P}} (|G| - 1) = |N_1| - |\mathscr{P}|. \quad \blacksquare$$

For the Sender to be persuasive against any (diverse) group of singled-minded Receivers, she must ensure that her statements have stakes by ensuring that her incentive to lie to a subset of the group is offset by her desire to be truthful to another subset of the group. Such effect wold not be present if the Sender can persuade all Receivers in the group to take action. Thus, partitioning the set of Receivers N_1 into smaller groups limits the maximum potential payoff from the Sender.

3.1 Solving for optimal partitions

In general, there are multiple partitions that can attain W^*. Example 2 demonstrates one source of multiplicity. In that example, both partitions $\{\{1,2\},\{3,4\}\}$ and $\{\{1,4\},\{2,3\}\}$ yield the Sender-optimal payoff of two. The multiplicity there arises from the fact that

Receivers 2 and 4 are identical—in particular, this means that they are interchangeable in any group that contains just one of them. Thus, from the Sender's perspective, the two agents are identical. Formally, say that Receivers $i,j \in N$ are of the same *type* if

$$w(G \cup \{i\}) = w(G \cup \{j\}) \; \forall G \subseteq N \backslash \{i,j\}.$$

Note that the definition allows preferences for Receivers of the same type to differ.

Following Aziz and Keijzer (2011) and Ueda et al. (2011), let T be the set of types of Receivers, $t_i \in T$ denote Receiver i's type, and n_t denote the number of type-t Receivers in N (i.e., $n_t := |\{i \in N : t_i = t\}|$). By definition, any $G, G' \subseteq N$ that contain the same number of types of Receivers must yield the same payoff. The Sender's problem can thus be stated equivalently as the problem of finding optimal ways to group the different types of Receivers. Towards this goal, given any $G \subseteq N$, let ψ^G be a vector that specifies the number of types of each Receiver that G contains; i.e.,

$$\psi^G = (|i \in G : t_i = t|)_{t \in T} \in \Psi := \times_{t \in T} \{0, \ldots, n_t\};$$

and define $\omega : \Psi \to \mathbb{R}$ by setting $\omega(\psi) = w(G)$ for any $G \subseteq N$ such that $\psi = \psi^G$. I refer to $\psi \in \Psi$ as a *type-group*. Notice that any partition $\mathscr{P} \in \Pi(N)$ can be represented as a collection of type-groups $\{\psi^G\}_{G \in \mathscr{P}} \subseteq \Psi$ such that

$$\sum_{G \in \mathscr{P}} \psi^G = (n_t)_{t \in T}.$$

The Sender's problem can now be stated as a problem of finding an optimal collection of type-groups; i.e.,

$$W^* = \max_{\{\psi\} \subseteq \Psi} \sum_{\psi' \in \{\psi\}} \omega\left(\psi'\right) \text{ s.t.} \sum_{\psi' \in \{\psi\}} \psi' = (n_t)_{t \in T}, \tag{7}$$

where, with slight abuse of notation, I let $\{\psi\}$ denote a collection of subsets of Ψ and $\psi \in \{\psi\}$ denote an element in the collection. In Example 2, letting $T = \Theta$, then W^* is attained by the unique type-groups $\{(1,1,0),(0,1,1)\}$.

Multiplicity can also arise from the fact N may contain multiple instances of type-groups. For example, if we add copies of Receivers in Example 2, then a type-group $(2,2,0)$ yields the same payoff as two of type-group $(1,1,0)$. To deal with this kind of multiplicity, I say that a type-group $\psi \in \Psi$ is *simple* if either (i) partitioning the group further leads to strictly lower payoffs,

$$\omega\left(\psi\right) > \max_{\{\psi'\}\subseteq\Psi,\ \{\psi'\}\neq\{\psi\}} \sum_{\psi''\in\{\psi\}} \omega\left(\psi''\right) \text{ s.t.} \sum_{\psi''\in\{\psi'\}} \psi'' = \psi;$$

or (ii) it is a singleton group (i.e., ψ is everywhere zero except for one type).[28]

Lemma 5. *Any optimal collection of type-groups can be expressed as a collection of simple type-groups.*

Proof. Fix a collection of type-groups $\{\psi^*\} \subseteq \Psi$ that is optimal but contains a type-group $\psi^* \in \{\psi^*\}$ that is not simple. By definition, there exists $\{\psi'\}$ such that $\sum_{\psi''\in\{\psi'\}} \psi'' = \psi^*$ with $|\{\psi'\}| > 1$ and $\sum_{\psi''\in\{\psi'\}} \omega(\psi'') \leq \omega(\psi^*)$. But since $\{\psi^*\}$ is optimal, the latter inequality must be an equality. ■

Let Ψ^* denote the set of simple groups such that $\omega(\psi) > 0$ if and only if $\psi \in \Psi^*$. The lemma above implies that we may replace Ψ with Ψ^* in (7).

The Sender's problem can be thought of as deciding on which and how many simple type-groups to form while ensuring that the total number of Receivers of each type does not exceed the available number of Receivers. Thus, the Sender's problem can be formulated as a multi-dimensional knapsack problem:

$$W^* = \max_{\left(x_\psi\right)\in\mathbb{Z}_+^{\Psi^*}} \sum_{\psi\in\Psi^*} \omega\left(\psi\right) x_\psi \text{ s.t.} \sum_{\psi\in\Psi^*} \psi x_\psi \leq \left(n_t\right)_{t\in T}.$$

The formulation above allows one to adopt algorithms that solve multi-dimensional knapsack problems to solve (possibly approximately) the Sender's problem.

Aziz and Keijzer (2011) show that it is also possible to solve for an optimal partition recursively. To that end, define $W : \Psi \to \mathbb{R}$ recursively as follows:

$$W\left(\psi\right) = \begin{cases} 0 & \text{if } \psi = \vec{0} \\ \max\left\{W\left(\left(n_t - x_t\right)_{t\in T}\right) + \omega\left(\left(x_t\right)_{t\in T}\right) : x_t \in \{0,\ldots,n_t\}\ \forall t \in T\right\} & \text{otherwise} \end{cases}.$$

The iterative procedure is as follows. First, start with $\psi_{\vec{0}} = \vec{0}$ and set $W(\vec{0}) = 0$, where $\vec{0}$ is a vector of $|T|$-many zeros. Next, consider groups containing one Receiver, $\psi' \in$

[28] Conitzer and Sandholm (2006) study CSG problems in which some groups of agents are not valuable.

$\{(1,0,\ldots),(0,1,0,\ldots),\ldots,(0,\ldots,0,1)\}$, and let

$$W\left(\psi'\right) := \max\left\{W\left(\vec{0}\right),\ W\left(\vec{0}\right)+\omega\left(\psi'\right)\right\}.$$

Next, consider groups containing two receivers, $\psi' \in \{(1,1,0,\ldots),(0,1,1,0,\ldots),\cdots\}$ and set

$$W(1,1,0,\ldots) = \max\left\{W(\vec{0})+\omega(1,1,0,\ldots), W(1,0,\ldots)+\omega(0,1,0\ldots),\cdots\right\},$$

and so on. Proceeding in this way gives $W((n_t)_{t\in T}) = W^*$.[29]

3.2 Properties of optimal communications

The following result shows the sense in which it is optimal for the Sender to focus on trying to persuade Receivers who are easier to persuade (i.e., those who are less extreme).

Proposition 2. *Suppose Receivers are single-minded. Then, there exists an optimal partition such that Receivers who are in groups that the Sender has a chance of persuading are all less extreme than those who are in groups that the Sender does not have a chance of persuading. That is, there exists $\mathscr{P}_1^*$ that attains $W^*(N_1)$ such that any i in some $G \in \mathscr{P}_1^*$ such that $w(G) = 0$ is more extreme than all i' with $\theta_i = \theta_{i'}$ in some $G' \in \mathscr{P}_1^*$ such that $w(G') > 0$.*

Proof. Let $\mathscr{P}_1^*$ be an optimal partition of N_1. Suppose that there exists i that belongs in some group $G \in \mathscr{P}_1^*$ such that $w(G) = 0$. Suppose further that there exists i' with $\theta_i = \theta_{i'}$ who belongs in some group $G' \in \mathscr{P}_1^*$ such that $w(G') > 0$ and i' is less extreme than i. Then, $B_j \subseteq B_i$. Thus, part (vi) of Lemma 4 implies that i and i' can be exchanged without affecting payoffs. Exchanging all such i with i' gives the desired result. ∎

Due to the combinatorial nature of the problem, deriving further properties of optimal communications is difficult even when restricting attention to single-minded Receivers. To derive more properties, I impose further restrictions on the Receiver's preferences. To that end, say that Receivers in N_1 are *contentious* if no single argument (i.e., posterior belief) is able to persuade Receivers who have differing opinions; i.e., $B_i \cap B_j \neq \varnothing$ for all $i, j \in N_1$ such that $\theta_i \neq \theta_j$. For example, Receivers in Figure 3a are contentious while Receivers in

[29]The algorithm runs in time $O(|N|^{2|T|})$.

Figure 3b are not contentious. Moreover, because, for any Receivers $i, j \in N_1$ with differing opinions (i.e., $\theta_i \neq \theta_j$),

$$B_i \cap B_j \neq \varnothing \Leftrightarrow \gamma_i + \gamma_j > 1.$$

Hence, Receivers are continuous when their opinions are sufficiently extreme.

Proposition 3. *Suppose that Receivers in N_1 are single-minded and contentious. Then, there exists an optimal partition of N_1 in which each group is simple and contains at most one Receiver of any particular opinion, and the Sender attains a payoff of at most one from each group; i.e., there exists $\mathscr{P}_1^* \in \Pi(N_1)$ such that $W^*(N_1) = \sum_{G \in \mathscr{P}_1^*} w(G)$, $|\{i \in G : \theta_i = \theta\}| \in \{0,1\}$ for all $\theta \in \Theta$, and for all $G \in \mathscr{P}^*$.*

Proof. Fix an optimal partition of $\mathscr{P}_1^* \in \Pi(N_1)$. Fix any $G \in \mathscr{P}_1^*$. By Lemma 3, there exist $\{\mu_k\}_{k=1}^K$ and $\{F_1, \ldots, F_K\} \subseteq G$ with $|F_k| = w(G)$ for all $k \in \{1, 2, \ldots, K\}$ such that μ_0 is a convex combination of $\{\mu_k\}_{k=1}^K$ such that $\mu_k \in B_{F_k}$ for all $k \in \{1, 2, \ldots, K\}$. Observe that, given any $i, j \in N_1$ such that $\theta_i \neq \theta_j$, $\gamma_i + \gamma_j > 1$ if and only if $B_i \cap B_j = \varnothing$. Thus, each F_k must contain $w(G)$ Receivers with the same opinion. Let $G_1 \subseteq P$ be a group that consists of one Receiver i_k from each F_k. Then, By Lemma 3, $w(G_1) = 1$ because μ_0 is a convex combination of $(\mu_k)_{k=1}^K$ such that $\mu_k \in B_{i_k}$for each $k \in \{1, 2, \ldots, K\}$. With the remaining Receivers, $G \backslash G_1$, construct G_2 consisting again of one Receiver from each $F_1 \backslash G_1, F_2 \backslash G_1, \ldots, F_K \backslash G_1$. Proceeding in this manner gives $\{G_k\}_{k=1}^{w(G)}$ such that $w(G_k) = 1$ for each $k \in \{1, \ldots, w(G)\}$. Since $w(\cdot) \geq 0$,

$$w(G) \leq \underbrace{\sum_{k=1}^{w(G)} w(G_k)}_{=w(G)} + w\left(G \backslash \bigcup_{k=1}^{w(G)} G_k\right).$$

However, because $(\{G_k\}_{k=1}^{w(G)}) \cup (G \backslash \bigcup_{k=1}^{w(G)} G_k)$ is a partition of G, the right-hand side must be less than $w(G)$ by optimality of G so that $w(G \backslash \bigcup_{k=1}^{w(G)} G_k) = 0$. Hence, by part (ii) of Lemma (2), the Sender cannot attain a strictly positive payoff from any subset of $G \backslash \bigcup_{k=1}^{w(G)} G_k$. Thus, I can partition $G \backslash \bigcup_{k=1}^{w(G)} G_k$ such that each (sub)group contains (at most) one Receiver with a particular opinion. ■

The contrapositive of Proposition 3 says the following: If the Sender is able to persuade two or more Receivers in a simple group,[30] then there must be an argument (i.e., a posterior

[30]For example, in Figure 3a, the Sender is able to always obtain two votes by the partition $\{\{1,2\},\{3,4\}\}$

belief) that can persuade Receivers with differing opinions to take action. In other words, the Sender is able to persuade two or more Receivers in a group in a "nontrivial manner" only when at least some Receivers' opinions are not too extreme (as in Figure 3b). In this sense, the Sender is able to benefit more from Receivers whose opinions are not extreme by grouping such Receivers together. The flip-side of this observation is that, when Receivers are contentious, the sender's communication is never ambiguous meaning that any posterior belief the Sender induces only persuades Receivers of one particular opinion.

Proposition 3 is useful when solving for the optimal partition because it means that one need not consider groups that contain more than $|\Theta|$ Receivers nor groups that contain more than one Receiver of any particular opinion. It turns out that, even if Receivers are not contentious, if $|\Theta| \in \{2,3\}$, it remains the case that an optimal partition exists in which no groups contain more than $|\Theta|$ Receivers and each group contains at most one Receiver of any particular opinion.[31]

Notice that when $|\Theta| = 3$, there are only three ways in which the Sender can secure a strictly positive payoff from a group containing up to three Receivers. The Sender can secure a payoff of: two from a trio (as in Figure 3b), one from a pair (as in Example 2), one from a trio (as in Example 1). Observe that these three cases can be ordered by their per-Receiver payoff for the Sender. This suggests that an optimal partition can be attained by an algorithm that sequentially maximises the number of groups that secure the highest per-Receiver payoff; i.e., first maximise the number of trios that can secure a payoff of two, then with the remaining Receivers, maximise the number of pairs that can secure a payoff of one, and finally, maximise the number of trios that can secure a payoff of one with the still remaining Receivers. I call such an algorithm the *greedy algorithm*. The following gives two sufficient conditions on the Receivers' preferences that ensure that the greedy algorithm attains an optimal communication. To state it, I say that Receivers are *homogenous* if Receivers with the same opinion have the same strength of opinion; i.e., for each $\theta \in \Theta$, there exists $\gamma_\theta \in (0,1)$ such that $\gamma_\theta = \gamma_i$ for all $i \in N_1$ such that $\theta_i = \theta$.

Proposition 4. *Suppose $|\Theta| \in \{2,3\}$ and Receivers are single-minded. The greedy algorithm yields an optimal partition if either Receivers are contentious or if Receivers are homogeneous.*

of Receivers (i.e., $\{1,2,3,4\}$ is not a simple group), whereas in Figure 3b, any nontrivial partition of Receivers would only guarantee one vote for the Sender (i.e., $\{1,2,3\}$ is a simple group).

[31] In a working version of the paper, I give an example with $|\Theta| = 4$ and non-contentious Receivers in which the maximum size of groups is strictly greater than 4 and groups might contain more than one Receiver of the same type.

Proof. See Appendix A.1. ∎

Recall that if Receivers are contentious, then no simple group can secure a payoff of two. Hence, optimal communication when $|\Theta| = 3$ involves either a trio or a pair that secure a payoff of one. The result above tells us that the Sender should first maximise the number of such pairs over trios. If Receivers are homogenous, then the only source of diversity in a group is the differences in the Receivers' opinions (i.e., it rules out Example 3).[32]

When the state is binary, Proposition 3 implies that it suffices to consider pairs of Receivers (who prefer opposite states). Thus, a partition that maximises the number of pairs that secure a payoff of one must be optimal. In other words, the greedy algorithm yields an optimal partition of N_1 when the state is binary (without any restrictions on the thresholds).

3.3 When is public communication sufficient?

Let us now consider when public communication is sufficient for the Sender. An immediate implication of the Corollary 1 is that public communication is the only way to attain a payoff $|N_1| - 1$ for the Sender. In other words, public communication is strictly preferred by the Sender over strictly semi-public communication if $|N_1| - 1$ is N_1-securable. The following establishes the conditions under which the Sender can attain $|N_1| - 1$ from the set N_1 of Receivers that require persuading.

Proposition 5. *Suppose Receivers are single-minded and $|N_1| \geq 3$ and that $w(N_1) = |N_1| - 1$ (i.e., $|N_1| - 1$ is N_1-securable), Then,*

(i) the size of the audience is smaller than the range of opinions (i.e., $|N_1| \leq |\Theta|$),

(ii) Receivers are maximally diverse (i.e., $\theta_i \neq \theta_j$ for all distinct $i, j \in N_1$),

(iii) Receivers are not contentious (i.e., it cannot be that $\gamma_i + \gamma_j > 1$ for all distinct $i, j \in N_1$).

Conversely, there exist $\mu_0 \in \Delta\Theta$ and a set N_1 of single-minded Receivers who require persuading (i.e., $\mu_0 \notin B_i$ for any $i \in N_1$) such that $w(N_1) = |N_1| - 1$ that satisfies properties (i) to (iii).

[32] In Appendix A.3, I give examples in which the greedy algorithm fails to attain an optimal partition because (i) $|\Theta| = 3$ but Receivers are not contentious, and (ii) Receivers are contentious and homogeneous but $|\Theta| = 4$.

Proof. Suppose $w(N_1) = |N_1| - 1$. Part (i) follows from part (v) of Lemma 4. By Lemma 3, there exists a collection of $L \leq |\Theta|$ subgroups of N_1, $\{G_\ell\}_{\ell=1}^{L}$, each consisting of exactly $|N_1| - 1$ Receivers from N_1 (i.e., $|G_\ell| = |N_1| - 1$) such that μ_0 is a convex combination of $\{\mu_\ell\}_{\ell=1}^{L}$ where $\mu_\ell \in B_{G_\ell}$ for all $\ell \in \{1, 2, \dots, L\}$. Since there are only $|N_1|$ Receivers, $L \leq |N_1|$. If $L < |N_1|$, then there would be at least one Receiver $i \in N_1$ that belongs in every subgroup which would imply that $\mu_0 \in B_i$ contradicting that $i \in N_1$. Thus, $L = |N_1|$ in which case, for each $\ell \in \{1, 2 \dots, L\}$, there exists a unique $i_\ell := N_1 \backslash G_\ell$. Suppose that two distinct Receivers $i, j \in N_1$ share an opinion; i.e., $\theta_i = \theta_j$. Without loss of generality, suppose $B_i \subseteq B_j$. Note that $\mu_j \in B_\ell$ for all $\ell \neq j$ and, in particular, $\mu_j \in B_i$. But then $\mu_j \in B_j$ because $B_i \subseteq B_j$. In particular, this means that $\mu_0 \in B_j$ contradicting that $j \in N_1$. It therefore follows that Receivers in N_1 must all have distinct opinions. Finally, suppose that $|N_1| \geq 3$ so that $|G_\ell| \geq 2$ for all $\ell \in \{1, 2, \dots, L\}$. Then, for any $i, j \in G_\ell$, it must be that $B_i \cap B_j \neq \varnothing$ requires that that $\gamma_i + \gamma_j \leq 1$.

To prove the converse, suppose $|N_1| = |\Theta|$ and that each Receiver in N_1 prefers distinct states and, abusing notation slightly, I let $N_1 = \Theta$. For each $\theta \in \Theta$, let $(\gamma_\theta)_{\theta \in \Theta}$ be such that $\gamma_\theta \in (\frac{1}{|\Theta|}, \frac{1}{|\Theta|-1})$ for all $\theta \in \Theta$. Observe that

$$\sum_{\tilde{\theta} \in \Theta \backslash \{\theta\}} \gamma_{\tilde{\theta}} < \sum_{\tilde{\theta} \in \Theta \backslash \{\theta\}} \frac{1}{|\Theta| - 1} = 1 \text{ and } \sum_{\tilde{\theta} \in \Theta} \gamma_{\tilde{\theta}} > \sum_{\tilde{\theta} \in \Theta} \frac{1}{|\Theta|} = 1.$$

The first inequality implies inequality implies that $B_{\Theta \backslash \{\theta\}} \neq \varnothing$ because, in particular, $\mu \in \Delta\Theta$ such that $\mu(\tilde{\theta}) = \gamma_{\tilde{\theta}}$ for all $\tilde{\theta} \in \Theta \backslash \{\theta\}$ and $\mu(\theta) = 1 - \sum_{\tilde{\theta} \in \Theta \backslash \{\theta\}} \gamma_\theta$ is contained in $B_{\Theta \backslash \{\theta\}}$. The second inequality implies that $\Delta\Theta \backslash \bigcup_{i \in G} B_i \neq \varnothing$ because any $\mu \in \Delta\Theta$ such that $\mu(\tilde{\theta}) < \gamma_{\tilde{\theta}}$ for all $\tilde{\theta} \in \Theta$ is contained in $\Delta\Theta \backslash \bigcup_{i \in G} B_i$. Thus, I can choose any $\mu_0 \in \Delta\Theta \backslash \bigcup_{i \in G} B_i$ as the prior belief and it would be that $w(N_1) = |N_1| - 1$. Observe that the construction above in fact is applicable when $|N_1| < |\Theta|$ if we choose $G = \tilde{\Theta}$ where $\tilde{\Theta} \subseteq \Theta$ with $|G| = |\tilde{\Theta}|$. ■

Recall that the benefit of strictly semi-public communication arises from the fact that the Sender can adopt different messaging strategies with respect to differently diverse groups of Receivers. Thus, public communication is optimal if the Sender is only able to persuade a group of a specific diversity. One example of such a case is when Receivers are extremely contentious so that the only way for the Sender to be able to persuade a Receiver in a group is if the group is maximally diverse (i.e., contains Receivers of every possible opinion).

Proposition 6. *Public communication is sufficient with respect to Receivers who are extremely contentious, and the Sender's payoff is given by the number of Receivers with the least popular opinion. That is, $\gamma_i \to 1$ for all $i \in N_1$, public communication with Receivers who require persuading is optimal and*

$$W^*(N_1) = w(N_1) = \min_{\theta \in \Theta} |\{i \in N_1 : \theta_i = \theta\}|.$$

Proof. I first show that, as Receivers become extremely sceptical (i.e., $\gamma_i \to 1$ for all $i \in N_1$), optimal partitions consist of groups that contain Receivers of every possible opinion. Recall that $\mu_0 \in \Delta\Theta$ has full support. Hence, there exists sufficiently high $\gamma = (\gamma_i)_{i \in N_1}$ such that μ_0 can only be expressed as a combination of elements from $\{B_i\}_{i \in G}$ if $\bigcup_{i \in G} \theta_i = \Theta$. Since $B_i \cap B_j = \varnothing$ for any district $i, j \in N_1$ with $\theta_i \neq \theta_j$ for sufficiently large γ_i and γ_j, $\cap_{i \in G} B_i \neq \varnothing$ if and only if $G \subseteq N_1$ contains Receivers of the same opinion. Thus, it follows that a group $G \subseteq N_1$ can secure a positive payoff only if it contains Receivers of every possible opinion. Then, by Lemma 3, we have $w(G) = \min_{\theta \in \Theta} |\{i \in G : \theta_i = \theta\}|$.

Toward showing that public communication is optimal, let $\mathscr{P}_1^*$ be an optimal partition of N_1. If $|\mathscr{P}^*| = 1$, then there is nothing to show. Suppose instead that $|\mathscr{P}_1^*| > 1$. For any $G, G' \in \mathscr{P}^*$, a payoff of $w(G) + w(G')$ is $G \cup G'$-securable because there are $w(G) + w(G')$-many Receivers of each of the possible opinions in $G \cup G'$. If $w(G \cup G') > w(G) + w(G')$, then there exists $S \subset G$ and $S' \subset G'$ such that $S \cup S'$ contains $w(G \cup G') - w(G) - w(G')$-many Receivers of every possible opinion. But this contradicts the fact that $\mathscr{P}_1^*$ is optimal since $(G \backslash S) \cup (G' \backslash S') \cup (S \cup S')$ is a partition of $G \cup G'$. Thus, $w(G \cup G') = w(G) + w(G')$ for any $G, G' \in \mathscr{P}_1^*$. It follows then that $w(\bigcup_{G \in \mathscr{P}^*} G) = w(\{N_1\}) = \min_{\theta \in \Theta} |\{i \in N_! : \theta_i = \theta\}$. ■

4 Discussion

I discuss a number of extensions and results under alternative assumptions below.

Commitment The desire for the Sender to communicate semi-publicly arises from the fact that Receivers do not inherently trust the Sender to communicate "truthfully". This lack of trust, formally captured as the Sender's inability to commit any communication strategy, means that the sender's communication is not credible unless it satisfies the incentive compatibility constraint, (2), in the definition of equilibrium. Semi-public communication benefits the Sender by giving more ways to satisfy the incentive compatibility constraint. If, instead, the Receivers trusted the Sender to communicate truthfully as in Bayesian persuasion (Kamenica and Gentzkow, 2011), the Sender's communication is credible even if it is not incentive compatible for the Sender. Consequently, there is no gain from communicating in groups and so private communication is always optimal for the Sender.

Signalling An assumption of the model is that the Sender chooses the partition of Receivers prior to observing the state. This assumption ensures that the Sender's choice of partition does not convey any information about the state. One may also consider the case in which the Sender chooses the partition *after* observing the state; however, allowing for such signalling does not affect the results. More concretely, in such a signalling version of the game, there always exists a pooling equilibrium in which the sender's payoff and the optimal partition (on the equilibrium path) correspond to the equilibrium payoff and partition in the original game.[33]

Cost of communicating in groups Recall that communication is costless in the model—in particular, not only are the messages payoff irrelevant, there are also no (marginal) costs associated with forming groups. While this is realistic in some situations (e.g., communication via emails), in other situations such as the sender communicating with receivers by holding meetings, it may be more plausible to include costs that depend on the number of groups in each partition. The costs of forming groups would be an additional force that pushes the sender to prefer public communication. Given that there are often many partitions that give rise to the same equilibrium payoff for the Sender, the costs of forming groups justifies the sender selecting the coarsest partition among optimal partitions.

Other Receiver preferences The single-minded Receiver preferences that I study above can be used as a building block for more general Receiver preferences. For example, if

[33]To see this, suppose toward a contradiction that there is a type $\theta \in \Theta$ that selects a different partition from all other types in equilibrium. Then, it must be that that type-θ Sender gets at least as high a payoff as others. If type-θ Sender gets a strictly higher payoff, then other types would deviate and choose the same partition as type-θ. Hence, if there was a separating/hybrid equilibrium in the signalling game, there must also exist a pooling equilibrium in which type-θ Sender selects the same partition as all other types. Moreover, any deviation from the on-path partition can be punished by an off-path belief that assumes any communication by the Sender to be uninformative.

a particular Receiver's actions are worth twice as much to the Sender than other Receivers, one could treat such a Receiver as a group of two "normal" Receivers. In solving for optimal partitions (section 3.1), one can then these two Receivers as a separate type-group. The same method can also account for the case when some Receivers' actions are complementary.

One can also derive properties of optimal partitions in case Receivers' preferences are "spatial" which can capture the situation in which, for example, the politician and voters are all either left- or right-wing, and each voter would vote for the politician only if their expectation of the politician's position on the political spectrum (represented as an interval) is sufficiently close to their own. Concretely, suppose that state space is the unit interval, $\Theta := [0,1]$,[34] and that the state $\theta \in \Theta$ represents the Sender's position on the spectrum; with $\theta = 0$ representing the "left" end of the spectrum and $\theta = 1$ representing the "right" end of the spectrum. For simplicity, suppose that the common prior belief $\mu_0 \in \Delta\Theta$ is atomless.[35] Each Receiver $i \in N$ takes action (i.e., $a_i = 1$) if he believes that the Sender's expected position on the spectrum is sufficiently close to his opinion $\theta_i \in \{0,1\}$; otherwise, the Receiver does not take action (i.e., $a_i = 0$). Each Receiver i's preference can be described by a pair $t_i = (\theta_i, \gamma_i) \in \{0,1\} \times [0,1]$, where i's payoff is given by

$$u_i(a_i, \theta) = (-1)^{\theta_i} a_i (\gamma_i - \theta).$$

Given two distinct Receivers $i, j \in N$ that prefer the same extreme (i.e., $\theta_i = \theta_j$), say that Receiver i is *less extreme* than Receiver j if Receiver i would take action whenever Receiver j would; i.e., $|\theta_i - \gamma_i| \geq |\theta_j - \gamma_j|$. The following characterises an optimal partition under this alternative preferences of Receivers.

Proposition 7. *There exists an optimal partition with the following properties.*

(i) Every Receiver is either paired with another Receiver or unpaired;

(ii) Unpaired Receivers are more extreme than any paired Receivers;

(iii) Receivers are paired negatively assortatively; i.e., among Receivers in pairs, the most extreme Receiver with ideal 0 *(or* 1*) is paired with the least extreme Receiver with opinion* 1 *(or* 0*), and so on.*

[34] Assuming that Θ is the unit interval is a normalisation.
[35] The results would not change materially even if the prior belief contained atoms.

Proof. See Appendix A.1. ■

The proposition shows that an optimal semi-public communication is one in which the partition consists of pairs and/or singletons of receivers in which pairs consist of receivers who prefer the opposite ends of the spectrum, and those in pairs are more moderate than those who are not paired. Communicating with groups that contain opposites is optimal for the sender because only the presence of receivers with the opposite preference would have a self-disciplining effect on the sender. While the above result also holds in the case of single-minded Receivers and two states, the result is less trivial in this case because each Receiver's optimal action depends on the mean (as opposed to belief) because not all Bayes plausible posterior distribution of means is attainable using some communication strategy (Gentzkow and Kamenica, 2016).

Multiple partitions While I have focused on the case in which the Sender selects a single partition of Receivers, it is also possible that the Sender selects multiple partitions of Receivers and a messaging strategy for each partition. Such a generalisation allows the Sender, for example, to send both public and private messages to Receivers (Goltsman and Pavlov, 2011; Arieli and Babichenko, 2019; Mathevet, Perego and Taneva, 2020). Note that allowing for multiple partitions allows one to use single-minded Receivers as building blocks for "multi-minded" Receivers. For example, suppose that a Receiver takes action only if his belief that the state is either $\theta \in \Theta$ or $\theta' \in \Theta \backslash \{\theta\}$ with sufficiently high probability. But such a Receiver can be thought of as a group of two single-minded Receivers with opinions θ and θ' if we allow additional communication to the pair *after* communicating with this pair in concert with other receivers in a group. In Appendix A.3, I also provide an example in which the ability for the Sender to adopt two nontrivial partitions can strictly benefit the Sender.

Information Leaks Whether the Sender can adopt any semi-public communication—thus her ability to benefit from designing her audience— depends on the extent to which the Sender can control information leaks across groups. When leaks occur due to information sharing between Receivers who belong to different groups, the Sender's optimal communication can still be derived by adding a constraint requiring such Receivers to belong to the same group. Observe that the extreme case in which information to any one group is known to leak to all other groups is equivalent to the case in which the Sender is constrained to adopting public communication.

5 Conclusion

In this paper, I explore how a sender who lacks receivers' trust can benefit from a mode of communication that I call *semi-public* communication in which the sender partitions the receivers into groups, and communicates publicly within each group but privately across groups. The benefit of communicating in groups arises from the fact that cheap-talk communication can be credible in front of an audience with diverse opinions because the sender's incentive to lie to some members of the audience can be offset by her incentive to be truthful to the other members of the audience. Semi-public communication enables the sender to communicate more effectively than private or public communication because it allows the sender to tailor communication to differently diverse groups to maximise her benefit from gaining credibility from the groups.

In a canonical game of persuasion with multiple receivers, I show that it is optimal for the sender to separate her audience into two groups based on whether the sender needs to persuade the receiver in the first place. The sender can further benefit by partitioning the group consisting of those that need persuading and I provide various characterisations of optimal partitions under different assumptions on the receivers' preferences. A practical implication of my results is that, while there is no need to ensure diversity of opinions in political rallies that are held for the supporters, a politician can be more persuasive to swing voters by campaigning across multiple events in which the audience consists of groups of swing voters that care about different sets of issues.

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A Appendix

A.1 Omitted proofs

A.1.1 Proof of Lemma 1

Proof of Lemma 1. Toward proving the result, say that the Sender's messaging strategy σ is *conditionally independent* if there exists a collection if measurable maps $(\sigma_G : \Theta \to \Delta M)_{G\in\mathscr{P}}$ such that $\sigma(m|\cdot) = \prod_{G\in\mathscr{P}} \sigma_G(m_G|\cdot)$ for all $m \in M^{\mathscr{P}}$. Now fix a partition $\mathscr{P} \in \Pi(N)$ and a $\mathscr{P}$-equilibrium (σ, α, μ). I prove the result by showing that there exists a $\mathscr{P}$-equilibrium $(\tilde{\sigma}, \tilde{\alpha}, \tilde{\mu})$ that yields the same payoff for the Sender but $\tilde{\sigma}$ is conditionally independent. Given any group $G \subseteq N$, define $v_G : \times_{i\in G} A_i \to \mathbb{R}$ as $v_G(a_G) := \sum_{i\in G} v_i(a_i)$. For each $G \in \mathscr{P}$, define $\tilde{\sigma}_G : \Theta \to \Delta M$ as

$$\tilde{\sigma}_G(m_G|\cdot) := \sum_{m_{-G}\in M_{-G}} \sigma((m_G; m_{-G})|\cdot).$$

The Sender's payoff from $\mathscr{P}$-equilibrium (σ, α, μ) is given by

$$\begin{aligned}
&\int_\Theta \sum_{m\in M^{\mathscr{P}}} \sum_{a\in A} \left[\sum_{G\in\mathscr{P}} v_G(a_G)\right] \underbrace{\left[\prod_{G\in\mathscr{P}} \prod_{i\in G} \alpha_i(a_i|m_G)\right]}_{=\alpha(a|m)} \sigma(m|\theta)\, \mathrm{d}\mu_0(\theta) \\
&= \sum_{\theta\in\Theta} \sum_{m\in M^{\mathscr{P}}} \sum_{G\in\mathscr{P}} \sum_{a_G\in A_G} v_G(a_G) \underbrace{\alpha_G(a_G|m_G)}_{=:\prod_{i\in G}\alpha_i(a_i|m_G)} \sigma(m|\theta)\, \mathrm{d}\mu_0(\theta) \\
&= \sum_{\theta\in\Theta} \sum_{G\in\mathscr{P}} \sum_{m_G\in M} \sum_{a_G\in A_G} v_G(a_G)\, \alpha_G(a_G|m_G) \underbrace{\sum_{m_{-G}\in M_{-G}} \sigma((m_G; m_{-G})|\theta) \mu_0(\theta)}_{=\tilde{\sigma}_G(m_G|\theta)} \\
&= \sum_{\theta\in\Theta} \sum_{m\in M^{\mathscr{P}}} \sum_{G\in\mathscr{P}} \left[\sum_{a_G\in A_G} v_G(a_G)\, \alpha_G(a_G|m_G)\, \tilde{\sigma}_G(m_G|\theta)\right] \mu_0(\theta) \\
&= \sum_{\theta\in\Theta} \sum_{m\in M^{\mathscr{P}}} \sum_{a\in A} \left[\sum_{G\in\mathscr{P}} v_G(a_G)\right] \prod_{G\in\mathscr{P}} \left(\left[\prod_{i\in G} \alpha_i(a_i|m_G)\right] \tilde{\sigma}_G(m_G|\theta)\right) \mu_0(\theta) \\
&= \sum_{\theta\in\Theta} \sum_{m\in M^{\mathscr{P}}} \sum_{a\in A} \left[\sum_{G\in\mathscr{P}} v_G(a_G)\right] \left[\prod_{G\in\mathscr{P}} \prod_{i\in G} \alpha_i(a_i|m_G)\right] \left[\prod_{G\in\mathscr{P}} \tilde{\sigma}_G(m_G|\theta)\right] \mu_0(\theta).
\end{aligned}$$

Thus, a payoff equivalent $\mathscr{P}$-equilibrium $(\tilde{\sigma}, \alpha, \tilde{\mu})$ exists, where $\tilde{\sigma}$ is a conditionally independent messaging strategy given by

$$\tilde{\sigma}(m|\cdot) := \prod_{G \in \mathscr{P}} \tilde{\sigma}_G(m_G|\cdot) \ \forall m \in M^{\mathscr{P}}.$$

Observe that $\text{supp}(\tilde{\sigma}) = \text{supp}(\sigma)$, and for each $G \in \mathscr{P}$, for all $m_G \in M$ such that there exists $m_{-G} \in M_{-G}$ such that $(m_G; m_{-G}) = \text{supp}(\tilde{\sigma})$, we let

$$\tilde{\mu}_G(\cdot|m_G) := \frac{\sum_{m_{-G} \in M_{-G}} \tilde{\sigma}((m_G; m_{-G})|\cdot)\mu_0(\cdot)}{\sum_{\theta \in \Theta} \sum_{m_{-G} \in M_{-G}} \tilde{\sigma}((m_G; m_{-G})|\theta)\mu_0(\theta)} = \mu_G(\cdot|m_G),$$

and for any other m_G's, $\tilde{\mu}_G(\cdot|m_G) := \mu_G(\cdot|m_G)$. Observe that σ and $\tilde{\sigma}$ induces the same distribution of posterior beliefs for each group and thus α remains optimal for the receivers.

■

A.1.2 Proof of Lemma 2

Proof of Lemma 2. Parts (i) and (iii) were proved in the main body. I first prove part (v). Fix a non-singleton group $G \subseteq N_1$ and $k \in \{1 \ldots, |G|-1\}$ such that $w(G) = |G| - k$. Then, by Lemma 3, there exist $\{\mu_\ell\}_{\ell=1}^{|\Theta|} \subseteq \Delta\Theta$ such that $\mu_\ell \in \bigcap_{i \in F_\ell} B_i$ and $F_\ell \subsetneq G$ with $|F_\ell| = |G| - k$ and μ_0 is a convex combination $\{\mu_\ell\}_{\ell=1}^{|\Theta|}$. Now consider a grid in which columns represent Receivers in G and the rows represent each elements of $\{F_1, \ldots, F_{|\Theta|}\}$. If $i \in S_\ell$, then the coordinate (S_ℓ, i) is marked with ● and if not ○. Since $|F_\ell| = |G| - k$, each row must have exactly k-many ○'s—call this rule #1. Moreover, that $G \subsetneq N_1$ means that each column must have at least one ○—call this rule rule #2.[36] Proving the result is equivalent to showing whether it is possible to fill in a $|\Theta|$-by-$|G|$ grid while obeying rules #1 and #2. For example, if $|G| = 4$ and $w(G) = 3$ (with $|\Theta| = 3$ still), because each row must have exactly ○, there must be a column that does not have any ○, violating rule #2 (see Figure 5).

[36] In Appendix (A.2), I explain how this formulation corresponds to a bipartite graph of Receivers with particular properties.

Figure 5: $w(G) = 3$ with $|\Theta| = 3$ and $|G| = 4$.

$F_\ell \backslash i$	1	2	3	4
F_1	○	●	●	●
F_2	●	○	●	●
F_3	●	●	○	●

More generally, there are $k|\Theta|$ many ○'s that need to be placed on a $|\Theta|$-by-$|G|$ grid. A way to proceed is to place one ○ diagonally from F_1 to $F_{|\Theta|}$ (as in the figure above) and if there are ○'s that are "left over", then start filling in the remaining ○'s diagonally again from F_1 to $F_{|\Theta|}$, and so on. Proceeding in this manner, one can always fill up to $k|\Theta|$ columns. Hence, it must be that $|G| \leq k|\Theta|$ (any other way to fill in $|G| = k|\Theta|$ columns can also be rearranged by exchanging columns and/or rows such that ○ lies "on the diagonals"). It remains to argue that if $|G| < k|\Theta|$, $k|\Theta|$-many ○'s can be placed in the grid while satisfying rules #1 and #2. Placing ○ on the diagonals first ensures that each row has at least one ○ so that rule #2 is not violated. Then, making sure each row has exactly k-many ○ is possible since $|G||\Theta| - k|\Theta| = (|G| - k)|\Theta| \geq 0$.

(iv) I first argue that $w(G) \leq |G| - 1$ for any $G \subseteq N_1$. Fix some $G \subseteq N_1$. Since the Sender's maximal payoff from each Receiver is 1, $w(G) \leq |G|$ and parts (i) and (iii) together implies $w(G) \in \{0, 1, \ldots, |G| - 1\}$. From part (v), if $w(G) = |G| - k$ for some $k \in \{1, \ldots, |G| - 1\}$, then

$$|G| \leq k|\Theta| \Leftrightarrow \frac{w(G)}{|G|} = 1 - \frac{k}{|G|} \leq 1 - \frac{1}{|\Theta|} \Rightarrow w(G) \leq |G| \frac{|\Theta| - 1}{|\Theta|} = |G| - \frac{|G|}{|\Theta|}.$$

If $|G| \leq |\Theta| \Leftrightarrow \frac{|G|}{|\Theta|} \leq 1$, then the right-hand side would be greater than $|G| - 1$ and so the bound is given by $|G| - 1$. If $|G| > |\Theta| \Leftrightarrow \frac{|G|}{|\Theta|} > 1$, then the right-hand side is smaller than $|G| - 1$ so that $|G| \frac{|\Theta|-1}{|\Theta|}$ is the lower upper bound. The result follows once we account for the fact that $|G| \frac{|\Theta|-1}{|\Theta|}$ may not be an integer while $w(G)$ has to be an integer. ■

Remark 1. The upper bound from part (iv) of Lemma 4 is tight. That is, for any finite Θ with $|\Theta| \geq 2$, there exit an interior $\mu_0 \in \Delta\Theta$ and a finite set G of Receivers such that $\mu_0 \notin B_i$ for all $i \in G$ and that (5) holds with equality. Consider first the case in which $|G| \leq |\Theta|$. Suppose $|G| = |\Theta|$ and that each Receiver in G prefers distinct states. Thus, we can identify each Receiver $i \in G$ by their opinion θ_i; i.e., $G = \Theta$. For each $\theta \in \Theta$, let $(\gamma_\theta)_{\theta \in \Theta}$ be such that $\gamma_\theta \in (\frac{1}{|\Theta|}, \frac{1}{|\Theta|-1})$ for all $\theta \in \Theta$. Observe that

$$\sum_{\tilde{\theta}\in\Theta\setminus\{\theta\}} \gamma_{\tilde{\theta}} < \sum_{\tilde{\theta}\in\Theta\setminus\{\theta\}} \frac{1}{|\Theta|-1} = 1$$

and that

$$\sum_{\tilde{\theta}\in\Theta} \gamma_{\tilde{\theta}} > \sum_{\tilde{\theta}\in\Theta} \frac{1}{|\Theta|} = 1.$$

The first implies inequality implies that $B_{\Theta\setminus\{\theta\}} \neq \varnothing$ for all $\theta \in \Theta$ because, in particular, $\mu \in \Delta\Theta$ such that $\mu(\tilde{\theta}) = \gamma_{\tilde{\theta}}$ for all $\tilde{\theta} \in \Theta\setminus\{\theta\}$ and $\mu(\theta) = 1 - \sum_{\tilde{\theta}\in\Theta\setminus\{\theta\}} \gamma_\theta$ is contained in $B_{\Theta\setminus\{\theta\}}$. The second inequality implies that $\Delta\Theta\setminus\bigcup_{i\in G} B_i \neq \varnothing$ because any $\mu \in \Delta\Theta$ such that $\mu(\tilde{\theta}) < \gamma_{\tilde{\theta}}$ for all $\tilde{\theta} \in \Theta$ is contained in $\Delta\Theta\setminus\bigcup_{i\in G} B_i$. Thus, I can pick any $\mu_0 \in \Delta\Theta\setminus\bigcup_{i\in G} B_i$ as the prior belief and $w(G) = |G| - 1$. Now fix any $k \in \{2, 3, \ldots\}$ and let $|G| = k|\Theta|$ (which exists by following the proof of part (v) of Lemma 4). Note that $|G| > |\Theta|$. I will show that there exists $\mu_0 \in \Delta\Theta$ and $\{(\theta_i, \gamma_i)\}_{i\in G}$ such that $w(G) = |G|\frac{|\Theta|-1}{|\Theta|} = k(|\Theta| - 1)$. Let G_1 be a set of $|\Theta|$ Receivers constructed as before such that $w(G_1) = |\Theta| - 1$. Let G be the set of $k|\Theta|$ Receivers obtained by duplicating each Receiver in G_1 k times. Then, $w(G) = k(|\Theta| - 1)$ as desired.

Remark 2. The construction in the proof of part (iv) of Lemma 4 when $|G| > |\Theta|$ means that Receivers in a group $G \subseteq N$ can be partitioned into smaller groups to attain the same payoff collectively. However, this is not always possible as the following shows.

Claim 1. Suppose $|\Theta| \geq 4$ and let $k \in \{2, 3, \ldots, \lfloor \frac{(|\Theta|-1)^2 + \sqrt{(|\Theta|-1)^2+4}}{2} \rfloor\}$. Then, there exists $\mu_0 \in \Delta\Theta$ and a set N_1 of single-minded Receivers that require persuading such that $|N_1| = k(|\Theta| - k) + 1$, $w(N_1) = |N_1| - k$, and $w(G) = 0$ for all non-empty strict subset G of N_1. In particular, whenever $|\Theta| \geq 6$, there exists such a group with a maximum size greater than $|\Theta|$.

Proof. The proof is constructive. Fix $|\Theta| \geq 4$ and $k \in \{2, 3, \ldots, \lfloor \frac{(|\Theta|-1)^2 + \sqrt{(|\Theta|-1)^2+4}}{2} \rfloor\}$. I will construct a group of $|N_1| = k(|\Theta| - k) + 1$-many single-minded Receivers that require persuading with the desired properties. Note that the upper bound on k ensures that $w(N_1) \geq 0$ and also that $|N_1| \geq 1$. I need to show that there exist $\mu_0 \in \Delta\Theta$ and $((\theta_i, \gamma_i)) \in (\Theta \times (0,1))^{|N_1|}$ such that $w(N_1) = |N_1| - k$, and $w(G) = 0$ for all non-empty strict subset G of N_1. Enumerate Receivers in N_1 as $N_1 = \{1, \ldots, |N_1|\}$. Suppose that Receivers 1 to $|\Theta|$ have distinct opinions and that, for $i \in \{1, \ldots, k+1\}$, Receiver $|\Theta| + i$ is identical to Receiver

$i-(\lfloor\frac{i}{(|\Theta|-(k+1))}\rfloor-1)(|\Theta|-(k+1))$. Thus, we only need to define γ_i for $i\in\{1,\ldots,|\Theta|\}$. To that end,

- $\triangleright$ for each $i\in\{1,\ldots,|\Theta|-(k+1)\}$, let $\sum_{j\in\{1,\ldots|\Theta|\}\setminus\{i\}}\gamma_j\leq 1$ so that $\bigcap_{j\in\{1,\ldots,|\Theta|\}\setminus\{i\}}B_j=\varnothing$;
- $\triangleright$ for each $i\in\{|\Theta|-k,\ldots,|\Theta|\}$, $\sum_{j\in\{1,\ldots,|\Theta|-(k+1)\}}\gamma_j+\gamma_i+\sum_{j\in\{|\Theta|+1,\ldots,|N_1|\}}\gamma_j\leq 1$ so that $(\bigcap_{j\in\{1,\ldots,|\Theta|-(k+1)\}}B_j)\cap B_i\cap(\bigcap_{j\in\{|\Theta|+1,\ldots,|N_1|\}}B_j)\neq\varnothing$;
- $\triangleright$ $\sum_{i=1}^{|\Theta|}\gamma_i>1$ so that $\Delta\Theta\setminus\bigcup_{i=1}^{|\Theta|}B_i\neq\varnothing$.

Above implies that there exists $\mu_0\in\Delta\Theta$ that is a convex combination of all elements of $\{\mu_i\}_{i=1}^{|\Theta|}$, where

$$\mu_i\in\begin{cases}\bigcap_{j\in\{1,\ldots,|\Theta|\}\setminus\{i\}}B_j & \text{if } i\in\{1,\ldots,|\Theta|-(k-1)\}\\ \left(\bigcap_{j\in\{1,\ldots,|\Theta|-(k+1)\}}B_j\right)\cap B_i\cap\left(\bigcap_{j\in\Theta|+1,\ldots,|N_1|}B_j\right) & \text{if } i\in\{|\Theta|-k,\ldots,|\Theta|\}\end{cases}.$$

The figure below shows what the above construction implies graphically when $|\Theta|=7$ and $k=3$ (and so $|\Theta|-(k+1)=3$).

Figure 6: Construction when $|\Theta|=7$, $k=3$.

$F_\ell\setminus i$	1	2	4	3	5	6	7	8	9	10	11	12	13
F_1	○	●	●	●	●	●	●	○	●	●	○	●	●
F_2	●	○	●	●	●	●	●	●	○	●	●	○	●
F_3	●	●	○	●	●	●	●	●	●	○	●	●	○
F_4	●	●	●	●	○	○	○	●	●	●	●	●	●
F_5	●	●	●	○	●	○	○	●	●	●	●	●	●
F_6	●	●	●	○	○	●	○	●	●	●	●	●	●
F_7	●	●	●	○	○	○	●	●	●	●	●	●	●

By construction, $w(\{1,\ldots,|\Theta|\})=0$ because for any μ_i for $i\in\{1,\ldots,|\Theta|-(k+1)\}$ belongs in $|\Theta|-1$ many B_j's while μ_i for $i\in\{|\Theta|-k,\ldots,|\Theta|\}$ belongs in $|\Theta|-k<|\Theta|-1$ many B_j's. By part (ii) of Lemma 2, it follows that no subset of $\{1,\ldots,|\Theta|\}$ can secure a strictly positive payoff. Moreover, adding any strict subsets of Receivers $\{|\Theta|+1,\ldots,|G|\}$ to $\{1,\ldots,|\Theta|\}$ does not allow the Sender to secure a strictly positive payoff either as there is at least μ_i for some $i\in\{1,\ldots,|\Theta|-(k+1)\}$ that belongs in stricter greater number of B_j's than others. Thus, by Lemma part (iv) of Lemma 4, no (strict) partition of N_1 can lead to a strictly positive payoff for the Sender.

Finally, observe that $|N_1|$ is maximised at $k^* = \frac{|\Theta|-1}{2}$ which implies group size of $\frac{(|\Theta|-1)^2}{4}$. Since $|\Theta| \geq 3$, we must have

$$\frac{(|\Theta|-1)^2}{4} \geq |\Theta| \Leftrightarrow |\Theta| \geq 3+2\sqrt{2} \approx 5.8.$$

Note that $\frac{(|\Theta|-1)^2}{4} \geq 1$ if $|\Theta| \geq 4$. ■

A.1.3 Proof of Proposition 4

Let us first consider the case with homogenous Receivers.

Lemma 6. *Suppose Receivers are single-minded and homogenous, and* $\Theta = \{\theta_1, \theta_2, \theta_3\}$. *For any* $G \subseteq N_1$, *there exists a partition* $\mathscr{P}_G \in \Pi(G)$, *such that* $|F| \leq 3$ *for all* $F \in \mathscr{P}_G$, $\sum_{F\in\mathscr{P}_G} w(F) \geq w(F)$, *and* $\mathscr{P}_G$ *can be obtained by a greedy algorithm.*

Proof. Suppose Receivers are single-minded and $|\Theta| = 3$ and fix $G \subseteq N_1$ with $w_G \equiv w(G) > 0$. By Lemma 1, μ_0 is a convex combination of beliefs in $\{B_F\}_{F\in\mathscr{F}}$ associated with exactly (i) two or (ii) three elements from $\mathscr{F} := \{F \subseteq G : |S| = w_G\}$.

Case (i): $\mu_0 = \alpha\mu_{F_1} + (1-\alpha)\mu_{F_2}$ for some $\alpha \in (0,1)$ and $\mu_{F_r} \in B_{F_r}$ with $|F_r| = w_G$ and $F_r \subseteq G$ for each $r \in \{1,2\}$. By convexity of each B_i, B_{F_r} is also convex and so $F_1 \cap F_2 = \varnothing$ because $G \subseteq N_1$). It follows that $|G| \geq 2w_G$. Take any pair $\{f_1, f_2\}$ where for each $r \in \{1,2\}$, $f_r \in F_r$ and observe that $\mu_{F_r} \in B_r$. By Lemma 1, a payoff of one is $\{f_1, f_2\}$-securable. Since w_G-many such pairs can be created, partitioning of G into such pairs together yields a payoff of at least w_G to the Sender.

Case (ii): $\mu_0 = \alpha_1\mu_{F_1} + \alpha_2\mu_{F_2} + (1-\alpha_1-\alpha_2)\mu_{F_3}$ for some $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$ and $\mu_{F_r} \in B_{F_r}$ with $|F_r| = w_G$ and $F_r \subseteq G$ for each $r \in \{1,2,3\}$. Convexity of B_i and the fact that $G \subseteq N_1$ means $F_1 \cap F_2 \cap F_3 = \varnothing$, which, in turn, means that no F_r can contain all types of Receiver; i.e., F_r contains only one type of Receiver (if $\gamma_\theta + \gamma_{\theta'} > 1$ for all distinct $\theta, \theta' \in \Theta$) or contains at most two distinct types of Receivers, say $\theta, \theta' \in \Theta$ (if $\gamma_\theta + \gamma_{\theta'} \leq 1$).

Consider the following figure, where columns represent Receivers in G and the rows represent each F_r. If $i \in F_r$, then the coordinate is marked with ● and if not ○. Without loss of generality, assume that Receivers $\{1,\ldots,w_G\}$ belong in F_1 (note $|G| > w_G$).

	1	2	$\cdots$	w_G	w_G+1	$\cdots$	$\lvert G\rvert$
F_1	●	●	$\cdots$	●	○	$\cdots$	○
F_2							
F_3							

For each $i \in F_1$, the possibilities are: i belongs in just F_1 or i belongs in either $F_1 \cap F_2$ or $F_1 \cap F_3$. Reorder i's such that

	$\{i\}_{i=1}^{n_1}$	$\{i\}_{i=n_1+1}^{n_1+n_{12}}$	$\{i\}_{i=n_1+n_{12}+1}^{n_1+n_{12}+n_{13}=w_G}$	w_G+1	$\cdots$	$\lvert G\rvert$
F_1	●	●	●	○	$\cdots$	○
F_2	○	●	○			
F_3	○	○	●			

where n_r denotes the number of Receivers in F_r that belongs only in F_r and n_{rt} denote the number of Receivers in F_r that belongs in both F_r and F_t. Given this notation, there must be $w_G - n_{12}$ many Receivers in $\{w_G+1,\ldots,|G|\}$. Such Receivers can either belong in just F_2 (n_2 many of them) or in F_{23} (n_{23} many of them). Reorder $\{w_G+1,\ldots|G|\}$ so that

	$\{i\}_{i=1}^{n_1}$	$\{i\}_{i=n_1+1}^{n_1+n_{12}}$	$\{i\}_{i=n_1+n_{12}+1}^{n_1+n_{12}+n_{13}=w_G}$	$\{i\}_{i=w_G+1}^{w_G+n_2}$	$\{i\}_{i=w_G+n_2+1}^{w_G+n_2+n_{23}}$	$\cdots$	$\lvert G\rvert$
F_1	●	●	●	○	○	$\cdots$	○
F_2	○	●	○	●	●		
F_3	○	○	●	○	●		

Then, $i \in \{w_G+n_2+n_{23}+1,\ldots|G|\}$, can belong in just F_3; i.e.,

	$\{i\}_{i=1}^{n_1}$	$\{i\}_{i=n_1+1}^{n_1+n_{12}}$	$\{i\}_{i=n_1+n_{12}+1}^{n_1+n_{12}+n_{13}=w_G}$	$\{i\}_{i=w_G+1}^{w_G+n_2}$	$\{i\}_{i=w_G+n_2+1}^{w_G+n_2+n_{23}}$	$\{i\}_{i=w_G+n_2+n_{23}+1}^{\lvert G\rvert}$
F_1	●	●	●	○	○	○
F_2	○	●	○	●	●	○
F_3	○	○	●	○	●	●

Since w_G is G-securable, we must have

$$|F_1| = n_1 + n_{12} + n_{13} = w_G, \tag{8}$$

$$|F_2| = n_2 + n_{12} + n_{23} = w_G, \tag{9}$$

$$|F_3| = n_3 + n_{13} + n_{23} = w_G. \tag{10}$$

Define $F_{rt} := F_r \cap F_t$ (and so $n_{rt} = |F_{rt}|$). Consider first a trio consisting of one Receiver each from $a \in F_{12}$, $b \in F_{13}$ and $c \in F_{23}$. The trio $\{a,b,c\}$ secures a payoff of two since

$$\mu_0 = \alpha\mu_1 + \beta\mu_2 + (1-\alpha-\beta)\,\mu_3$$

for some $\mu_1 \in B_a \cap B_b$, $\mu_2 \in B_a \cap B_c$ and $\mu_3 \in B_b \cap B_c$. There can be at most $\min\{n_{12}, n_{13}, n_{23}\}$ many such trios. Without loss of generality, suppose that $\min\{n_{12}, n_{13}, n_{23}\} = n_{12}$ (i.e., $n_{13}, n_{23} \geq n_{12}$) and so there are $n_{13} - n_{12}$ and $n_{23} - n_{12}$ many Receivers left in F_{13} and F_{23}, respectively. Observe that $i \in F_{13}$ can be paired with $j \in F_2 \cup F_{23}$ to secure a payoff of one since $\mu_1, \mu_3 \in B_i$ and $\mu_2 \in B_j$. Similarly, $i \in F_{23}$ can be paired with $j \in F_1 \cup F_{13}$ to secure a payoff of one. The number of pairs that can be formed are as follows.

- ▷ $F_{13} \times F_2$: $\min\{n_{13} - n_{12}, n_2\}$.
- ▷ $F_{23} \times F_1$: $\min\{n_{23} - n_{12}, n_1\}$.
- ▷ $F_{13} \times F_{23}$: $\min\{n_{13} - n_{12}, n_{23} - n_{12}\}$.

Suppose that the Sender creates $\min\{n_{13} - n_{12}, n_2\}$ pairs of $F_{13} \times F_2$ first, followed by pairs of $F_{23} \times F_1$.

- ▷ If $\min\{n_{13} - n_{12}, n_2\} = n_{13} - n_{12}$, then there may be Receivers left over in F_2's but no more $j \in F_{13}$ to pair them with. However, there are still $n_{23} - n_{12}$ many Receivers left in F_{23} who can be paired with $j \in F_1$ to secure a payoff of one; $\min\{n_{23} - n_{12}, n_1\}$ many such pairs can be formed. Consider now the Sender's ability to produce other pairs of $F_{23} \times F_1$ that secure a payoff of one.
  - ▷ If $\min\{n_{23} - n_{12}, n_1\} = n_1$, then Sender can secure a total payoff of

    $$w(\mathscr{P}_G) = 2n_{12} + (n_{13} - n_{12}) + n_1 = w_G,$$

    where the last equality follows from (8).

▷ If $\min\{n_{23}-n_{12},n_1\}=n_{23}-n_{12}$, then there may be Receivers in F_1, F_2 and F_3—a trio consisting of each such Receiver can secure a payoff of one. Hence, by (8)–(10),

$$\begin{aligned} w(\mathscr{P}_G) &= 2n_{12}+(n_{13}-n_{12})+(n_{23}-n_{12})+\min\{n_1,n_2,n_3\} \\ &= \begin{cases} \underbrace{n_{12}+n_{13}+n_1}_{=w_G}+\underbrace{n_{23}-n_{12}}_{\geq 0} & \text{if } \min\{n_1,n_2,n_3\}=n_1 \\ \underbrace{n_{12}+n_{23}+n_2}_{=w_G}+\underbrace{n_{13}-n_{12}}_{\geq 0} & \text{if } \min\{n_1,n_2,n_3\}=n_2 \\ \underbrace{n_{13}+n_{23}+n_3}_{=w_G}+\underbrace{n_{12}-n_{12}}_{=0} & \text{if } \min\{n_1,n_2,n_3\}=n_3 \end{cases} \\ &\geq w_G. \end{aligned}$$

▷ If $\min\{n_{13}-n_{12},n_2\}=n_2$, then there may be Receivers left over in F_{13} but no more Receivers in F_2 to pair them with. However, they can be paired with Receivers left in F_{23} to secure a payoff of one; $\min\{n_{13}-n_{12}-n_2,n_{23}-n_{12}\}$ many such pairs can be formed.

▷ If $\min\{n_{13}-n_{12}-n_2,n_{23}-n_{12}\}=n_{23}-n_{12}$, then there are no Receivers that can be paired with Receivers left over in F_{13} to secure a payoff of one. Hence, by (10),

$$w(\mathscr{P}_G)=2n_{12}+n_2+n_{23}-n_{12}=n_{12}+n_2+n_{23}=w_G.$$

▷ If $\min\{n_{13}-n_{12}-n_2,n_{23}-n_{12}\}=n_{13}-n_{12}-n_2$, then there may be Receivers left over in F_{23} who can be paired with Receivers in F_1 to secure a payoff of one; $\min\{n_{23}-n_{12}-(n_{13}-n_{12}-n_2),n_1\}$ many such pairs can be formed.

- If $\min\{n_{23}-n_{12}-(n_{13}-n_{12}-n_2),n_1\}=n_1$, then, by (8),

$$w(\mathscr{P}_G)\geq 2n_{12}+n_2+(n_{13}-n_{12}-n_2)+n_1=n_{12}+n_{13}+n_1=w_G.$$

- If $\min\{n_{23}-n_{12}-(n_{13}-n_{12}-n_2),n_1\}=n_{23}-n_{12}-(n_{13}-n_{12}-n_2)$, then, by (9),

$$
\begin{aligned}
w(\mathscr{P}_G) &\geq 2n_{12} + n_2 + (n_{13} - n_{12} - n_2) + n_{23} - n_{12} - (n_{13} - n_{12} - n_2) \\
&= n_{12} + n_{23} + n_2 = w_G.
\end{aligned}
$$

Hence, we can construct a partition of G in which each group consists of no more than three Receivers that together can secure at least w_G.

Finally, observe that in both cases (i) and (ii), the partition of G can be attained via the greedy algorithm. ■

Proof of Proposition 4. First, Lemma 6 proves the result for the case in which Receivers are homogenous and $|\Theta| = 3$. If Receivers are instead contentious (but $|\Theta| = 3$), then observe that a trio can no longer secure a payoff of two (recall Proposition 5). Thus, the same proof as in Lemma 6 implies that the Sender should simply maximise the number of pairs that can secure a payoff of one. Finally, suppose $|\Theta| = 2$. Once again, only a pair can secure a payoff of two and the same argument as in the proof of Lemma 6 implies the greedy algorithm yields an optimal partition. ■

A.1.4 Proof of Proposition 7

The set of posterior beliefs under which Receiver i with spatial preferences takes action is given by

$$
B_i := \left\{ \mu \in \Delta\Theta : (-1)^{\theta_i} \left(\gamma_i - \mathbb{E}_\mu [\theta] \right) \geq 0 \right\}.
$$

Under this definition of B_i, the condition (4) for G-securability remains the same and Theorem (2) continues to hold so that one may focus attention on how the Sender should group Receivers that require persuading (i.e., N_1). Let $\tilde{F}_i : \Theta \rightrightarrows \mathbb{R}$ denote the Sender's value correspondence in terms of posterior means given by

$$
\tilde{F}_i(E) := \begin{cases} \{1\} & \text{if } |\theta_i - \gamma_i| > |\theta_i - E| \\ [0,1] & \text{if } |\theta_i - \gamma_i| = |\theta_i - E| \\ \{0\} & \text{if } |\theta_i - \gamma_i| < |\theta_i - E| \end{cases}.
$$

The following lemma gives a characterisation of G-securability when Receivers have spatial preferences.

Lemma 7. *When Receivers have spatial preferences, for any $G \subseteq N_1$, a payoff $s \in \mathbb{R}$ is G-securable if and only if there exists a cutoff $k \in [0,1]$ such that*

$$s \in \left[\sum_{i \in G} \tilde{F}_i \left(\mathbb{E}_{\mu_0}[\theta | \theta \geq k]\right)\right] \bigcap \left[\sum_{i \in G} \tilde{F}_i \left(\mathbb{E}_{\mu_0}[\theta | \theta \leq k]\right)\right]. \tag{11}$$

Proof. Given that $G \subseteq N_1$, $\max \tilde{F}_i$ is quasiconvex. The result is then immediate from Claim 5 in Lipnowski and Ravid (2020). ■

In words, it suffices for the the Sender tells a group of Receivers with spatial preferences whether θ is above or below some cutoff $k \in [0,1]$. Thus, the benefit of semi-public communication is that it allows the Sender to adopt different cutoffs for each group. The following lemma is the key to proving Proposition 7.

Lemma 8. *Suppose Receivers have spatial preferences. If a pair of Receivers secures a payoff of one, then the Sender can secure a payoff of one from a pair consisting of less extreme Receivers.*

Proof. For $i_0, i_1 \in N_1$ such that $w(\{i_0, i_1\}) = 1$ and that i_0 (resp. i_1)'s opinion is 0 (resp. 1). By Lemma 7, there exists a cutoff $k \in \Theta$ such that $\gamma_{i_0} \geq \mathbb{E}_{\mu_0}[\theta | \theta \leq k]$ and $\gamma_{i_1} \leq \mathbb{E}_{\mu_0}[\theta | \theta \geq k]$. Let $i'_0, i'_1 \in N_1$ be less extreme than i_0 and i_1 respectively. Then, the same k satisfies (11) so that by Lemma 7, $w(\{i'_0, i'_1\}) = 0$. ■

Proof of Proposition 7. The proof proceeds by showing that any optimal partition can be further partitioned to satisfy condition (i). I then show that condition (iii) can be satisfied among Receivers who are paired using Lemma 8. The same lemma then can then be used to show any unpaired Receivers who are less extreme than some paired Receivers can be exchanged to satisfy (ii).

Fix an optimal partition $\mathscr{P}^* \in \Pi(N_1)$ with $W(\mathscr{P}^*) = W^*$. I first construct a partition of N_1 from $\mathscr{P}^*$ that consists of pairs and singleton Receivers. I will then show that Receivers in pairs can be exchanged to ensure negative assortativity. To that end, for each group $G \in \mathscr{P}^*$ such that $|G| > 2$, there exists a cutoff $k_G \in \Theta$ that satisfies (11) with $w_G = w(G)$. For each $z \in \{0,1\}$, let $G^z \subseteq G$ be the Receivers of opinion z that take action $a = 1$under the cutoff k_G (note$|G^z| = w(G)$). Since a payoff of one is securable with any pair $(i_0, i_1) \in G^0 \times G^1$ with the same cutoff, $G^0 \cup G^1$ can be decomposed into pairs that each secure a payoff of one that altogether secure a payoff of $w(G)$. Let $\mathscr{P}_G$ denote the partition of G consisting of unions of such pairs of Receivers with singleton sets of Receivers in

$G \backslash \{G^0 \cup G^1\}$. Then, $\mathscr{P} := \bigcup_{G \in \mathscr{P}^*} \mathscr{P}_G$ is a partition of N consisting of pairs of Receivers that each secure a payoff of one and singleton sets of Receivers such that $W(\mathscr{P}) = W^*$.

Let $\{(i_0^1, i_1^1), (i_0^2, i_1^2), \ldots, (i_0^{W^*}, i_1^{W^*}\}$ denote the pairs in $\mathscr{P}$ such that i_0^j is more extreme than i_0^{j+1} for each $z \in \{0,1\}$ and each $j \in \{1, \ldots, W^* - 1\}$. Suppose that i_0^1 (i.e., the most extreme Receiver with opinion 0 among those paired) is not paired with the most moderate Receiver with opinion 1 among those paired (if not, then repeat the process for i_0^2); i.e., there exists $j \in \{2, \ldots, W^*\}$ such that i_1^j is strictly more moderate that i_1^1. Let $i_1^{j'}$ be the least moderate among all such j's. By Lemma 8, $i_1^{j'}$ and i_1^1 can be exchange without affecting Sender's payoff. Now repeat the process for i_0^2, and so on. The process clearly terminates. Let $\tilde{\mathscr{S}}$ denote the set of pairs after this process terminates and observe that the pairs satisfy property (iii) among all Receivers who are paired in $\tilde{S}$.

Now suppose there exists an unpaired Receiver with opinion $z \in \{0,1\}$, say i_z, who is more moderate than some Receiver with opinion z who is paired. By Lemma 8, such a Receiver can be exchanged with the most extreme Receiver with opinion z who is paired and is also more moderate than Receiver i_z without affecting Sender's payoff. Repeat this process until there are no such i_z's. Then, all the unpaired Receivers are more moderate than paired Receivers (with the same opinion). ■

A.2 Interpreting G-securability as a graph

Given a set $N = \{1, \ldots, n\}$ of Receivers, $k \in \{1, \ldots, |N|\}$ and $w \in \{1, 2, \ldots n\}$, let $\mathscr{F}(w)$ denote the collection of distinct subsets of N of cardinality w with no common element; i.e.,

$$\mathscr{F}(w) := \left\{ \mathscr{G} \in \bigcup \{F \subseteq N : |F| = w\} : \bigcap \{G \in \mathscr{G}\} = \varnothing \right\}.$$

Given $G \subseteq N_1$, as mentioned above in the proof of Lemma 4, a payoff $w \in \mathbb{Z}_+$ is G-securable if there exists a collection of posterior beliefs $\{\mu\} \subseteq \Delta\Theta$ such that $\mu \in \bigcap_{i \in F_\mu} B_i$ for some $F_\mu \in \mathscr{F}(w)$, and $\bigcap_\mu F_\mu = \varnothing$.

As an example, consider the case in which a payoff of 3 is G-securable with $G = \{1,2,3,4,5\}$ because there exits $\{\mu_a, \mu_b, \mu_c\}$ such that

$$\mu_a \in B_1 \cap B_2 \cap B_5,$$
$$\mu_b \in B_1 \cap B_3 \cap B_4,$$
$$\mu_c \in B_2 \cap B_3 \cap B_4 \cap B_5.$$

Using the notation developed in the proof of Lemma 4, the situation above can be expressed as below.

Figure 7: $w(G) = 3$ with $|G| = 5$.

$F_\ell \backslash i$	1	2	3	4	5
F_1	●	●	○	○	●
F_2	●	○	●	●	○
F_3	○	●	●	●	●

.

Observe that the figure above can also be expressed as a bipartite graph. To that end, recall that an (undirected) graph is a pair (V,E), where V is the set of vertices and E is the set of edges that connects two vertices such that a vertex $i \in V$ is connected to a vertex $j \in V$ if and only if $(i, j) \in E$ and $(j,i) \in E$. A *bipartite graph with parts* $\{G,F\}$, denoted (G,F,E), is a graph (V,E) such that: (i) $\{G,F\}$ forms a partition of V; and (ii) $E \subseteq (G \times F) \cup (F \times G)$. Then, a payoff $w \in \mathbb{Z}_+$ is G-securable if and only if there exists a bipartite graph $(G,F := \{1,\ldots,|\Theta|\},E)$ such that: (i) each vertex in G has at most $k-1$ edges i.e.,

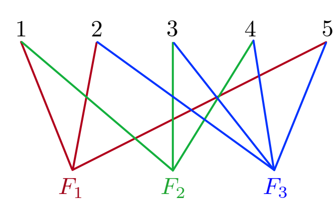
$$|\{j \in F : (i, j) \in E\}| \leq k-1 \ \forall i \in G;$$

and (ii) each vertex in F has at least w edges, i.e.,

$$|\{i \in N : (i, j) \in E\}| \geq w \ \forall j \in F.$$

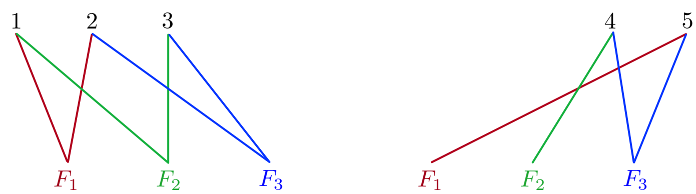
Thus, the example above can be described with the following bipartite graph.

Figure 8: Bipartite graph: $w(G) = 3$ with $|G| = 5$.



Whether a payoff of G can be further split without affecting payoffs depends on whether there exist subgraphs that together can secure a payoff of three. In this example, we can create two bipartite subgraphs by partitioning G as $\{\{1,2,3\},\{4,5\}\}$. Observe from the figure below that the subgraph consisting of $\{1,2,3\}$ secures a payoff of two (count the minimum number of edges from vertices in $\{S_r\}$) and the subgraph consisting of $\{4,5\}$ secures a payoff of one; i.e., together the two graphs secure a payoff of three.

Figure 9: 3 is securable with $\{\{a,b,c\},\{d,e\}\}$.



As noted previously, not all groups can be split without leading to lower overall payoffs.

A.3 Examples

A.3.1 Maximum group size

The example below shows that when $|\Theta| = 4$ and Receivers are not contentious, the maximum size of groups need not be less than or equal to $|\Theta|$ and groups might contain more than one Receiver of the same type.

Example 4. Suppose $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, $N = N_1 = \{1,2,3,4,5\}$ with

$$t_1 = t_5 = \theta_1,\ t_2 = \theta_2,\ t_3 = \theta_3,\ t_4 = \theta_4,$$
$$\gamma_{\theta_1} = \frac{3}{4},\ \gamma_{\theta_2} = \gamma_{\theta_3} = \gamma_{\theta_4} = \frac{1}{4},$$

and $\mu_0 = (\frac{1+\varepsilon}{2}, \frac{1-\varepsilon}{6}, \frac{1-\varepsilon}{6}, \frac{1-\varepsilon}{6})$ for some $\varepsilon > 0$ small. Then, $\mu_0(\theta) \leq \gamma_\theta$ for all $\theta \in \Theta$. Moreover, given

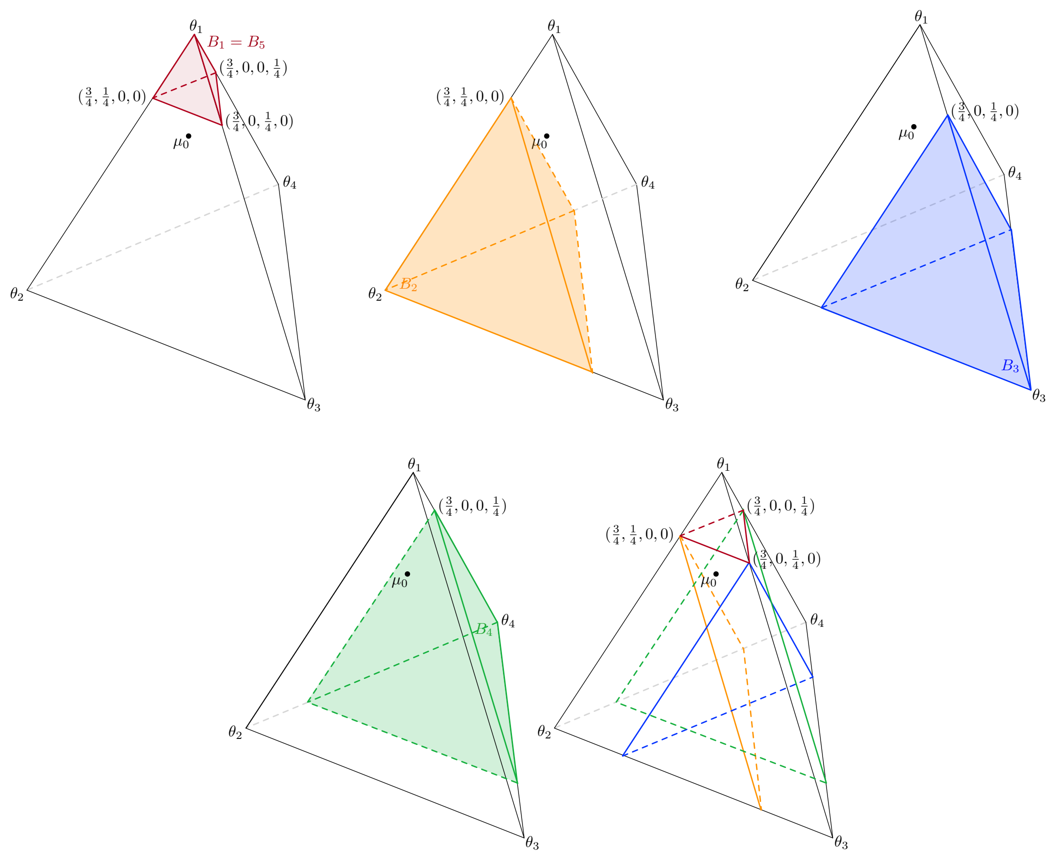
$$\mu_1 = \left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) \in B_2 \cap B_3 \cap B_4, \quad \mu_3 = \left(\frac{3}{4},0,\frac{1}{4},0\right) \in B_1 \cap B_3 \cap B_5,$$
$$\mu_2 = \left(\frac{3}{4},\frac{1}{4},0,0\right) \in B_1 \cap B_2 \cap B_5, \quad \mu_4 = \left(\frac{3}{4},0,0,\frac{1}{4}\right) \in B_1 \cap B_4 \cap B_5,$$

μ_0 can be expressed as

$$\mu_0 = \frac{3-2(1+\varepsilon)}{2}\mu_1 + \frac{1+2\varepsilon}{6}\mu_2 + \frac{1+2\varepsilon}{6}\mu_3 + \frac{1+2\varepsilon}{6}\mu_4.$$

Therefore, by Lemma 3, $w(N_1) = 3$. However, no (nontrivial) partition of N_1 can guarantee a payoff of 3 for the Sender. Since a partition of N_1 can secure 3 if and only if (i) a group consisting of four Receivers secures a payoff of three or (ii) a group consisting of three Receivers, say $P_1 \subseteq N_1$, secures a payoff of two, and another group consisting of two Receivers, say $P_2 = N_1 \backslash P_1$, secures a payoff of one.

Figure 10: Maximum size of groups need not be less than $|\Theta|$ when $|\Theta| = 4$.



(i) Note that $B_2 \cap B_3 \cap B_4$, $B_1 \cap B_2 \cap B_5 = \mu_2$, $B_1 \cap B_3 \cap B_5 = \mu_3$, $B_1 \cap B_4 \cap B_5 = \mu_4$ are the only (set of) beliefs in which three Receivers would take action $a = 1$. Removing Receiver 1 or 5 from N would leave only $B_2 \cap B_3 \cap B_4$ so that, given $\mu_0 \notin B_i$ for all $i \in N$, the remainder of Receivers cannot secure a payoff of one for the Sender. Consider removing Receiver 4 (the argument is symmetric for Receivers 2 and 3), which leaves $B_1 \cap B_2 \cap B_5 = \mu_2$ and $B_1 \cap B_3 \cap B_5 = \mu_3$. However, for any $\alpha \in [0,1]$,

$$\alpha \mu_2 + (1-\alpha)\mu_3 = \left(\frac{3}{4}, \frac{\alpha}{4}, \frac{1-\alpha}{4}, 0\right)$$

so that $\mu_0 \notin \text{co}(\{\mu_2, \mu_3\})$. Hence, by Lemma 3, $\{1,2,3,5\}$ cannot secure a payoff of three. It follows that the group of four Receivers cannot secure a payoff of three for the Sender.

(ii) Since $P \subseteq N_1$, $P_2 \neq \{1,5\}$. Consider two cases: (a) P_2 consists of type-θ_1 Receiver and a type-θ_2 Receiver (symmetric argument for θ_3 and θ_4) or (b) P_2 consists of type-θ_2 and type-θ_3 Receivers (symmetric argument in the case P_2 consists of types $\{\theta_2,\theta_4\}$ or $\{\theta_3,\theta_4\}$).

(a) In the first case, $P_1 = \{1,3,4\}$ and $P_2 = \{2,5\}$. The set of beliefs under which two Receivers in P_1 take action $a = 1$ are: $B_3 \cap B_4$, $B_1 \cap B_3 = \mu_3$ and $B_1 \cap B_4 = \mu_4$. Suppose there exists $\mu_{34} \in B_3 \cap B_4$ such that

$$\begin{aligned}
\mu_0(\theta_1) &= (1-\alpha-\beta)\,\mu_{34}(\theta_1) + \alpha\mu_3(\theta_1) + \beta\mu_4(\theta_1) \\
&= (1-\alpha-\beta)\,\mu_{34}(\theta_1) + \frac{3}{4}(\alpha+\beta), \\
\mu_0(\theta_2) &= (1-\alpha-\beta)\,\mu_{34}(\theta_2) + \alpha\mu_3(\theta_2) + \beta\mu_4(\theta_2) \\
&= (1-\alpha-\beta)\,\mu_{34}(\theta_2). \qquad (12)
\end{aligned}$$

Adding the two together gives

$$\mu_0(\theta_1) + \mu_0(\theta_2) = \frac{3}{4}(\alpha+\beta) + (1-\alpha-\beta)\left(\mu_{34}(\theta_1) + \mu_{34}(\theta_2)\right).$$

Since $\mu_{34} \in B_3 \cap B_4$, $\mu_{34}(\theta_1) + \mu_{34}(\theta_2) \leq 1 - \gamma_{\theta_3} - \gamma_{\theta_4} = \frac{1}{2}$ so that

$$\mu_0(\theta_1) + \mu_0(\theta_2) \leq \frac{3}{4}(\alpha+\beta) + \frac{1}{2}(1-\alpha-\beta) = \frac{1}{2} + \frac{1}{4}(\alpha+\beta).$$

Using (12) and the fact that $\mu_{34}(\theta_2) \leq \mu_{34}(\theta_1) + \mu_{34}(\theta_2) \leq \frac{1}{2}$ gives

$$\alpha + \beta = 1 - \frac{\mu_0(\theta_2)}{\mu_{34}(\theta_2)} \leq 1 - 2\mu_0(\theta_2)$$

$$\alpha + \beta = 1 - \frac{\mu_0(\theta_2)}{\mu_{34}(\theta_2)} \in \left[\mu_0(\theta_2),\, 1 - 2\mu_0(\theta_2)\right]$$

so that

$$\mu_0(\theta_1)+\mu_0(\theta_2)\leq \frac{1}{2}+\frac{1}{4}(1-2\mu_0(\theta_2))=\frac{3}{4}-\frac{1}{2}\mu_0(\theta_2)$$
$$\Leftrightarrow \mu_0(\theta_1)\leq \frac{3}{4}-\frac{3}{2}\mu_0(\theta_2).$$

Substituting the values for $\mu_0(\theta_1)$ and $\mu_0(\theta_2)$ gives

$$\frac{1+\varepsilon}{2}\leq \frac{3}{4}-\frac{3}{2}\left(\frac{1-\varepsilon}{6}\right)=\frac{1}{4}(2+\varepsilon)\Leftrightarrow \varepsilon\leq 0,$$

which contradicts the assumption that $\varepsilon > 0$. Hence, 2 is not P_1-securable.

(b) In the second case, $P_1 = \{1,4,5\}$. However, since $B_1 = B_5$, that $P \subseteq N_1$ implies that $w(P_1) = 1$. Hence, 2 is not P_1-securable.

□

A.3.2 Failures of the greedy algorithm

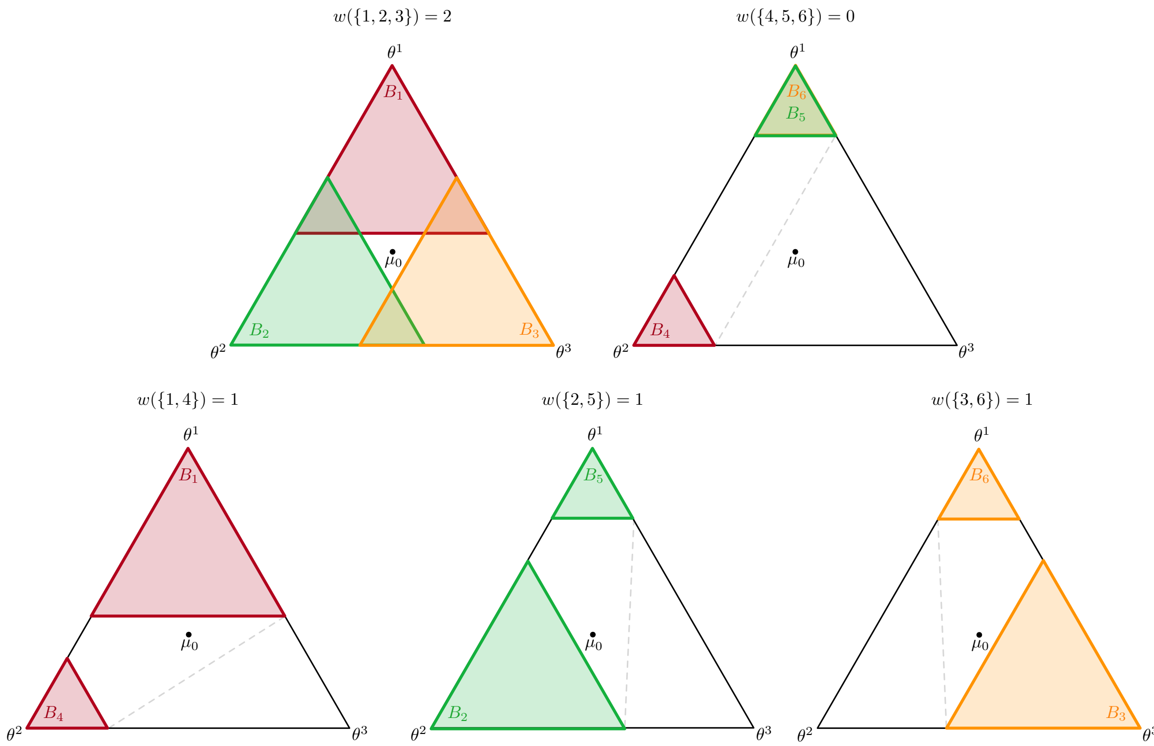
The next example shows that the greedy algorithm fails to attain an optimal communication when Receivers are not contentious even when $|\Theta| = 3$.

Example 5. Suppose $\Theta = \{\theta^1,\theta^2,\theta^3\}$ and that $N = \{1,2,3,4,5,6\}$. Receivers are all single-minded and their preferences are:

$$((\theta_i,\gamma_i))_{i=1}^{6}=\left(\left(\theta^1,\frac{2}{5}\right),\left(\theta^2,\frac{2}{5}\right),\left(\theta^1,\frac{2}{5}\right),\left(\theta^2,\frac{3}{4}\right),\left(\theta^1,\frac{3}{4}\right),\left(\theta^1,\frac{3}{4}\right)\right).$$

Assume that $\mu_0 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and observe that $\mu_0 \notin B_i$ for all $i \in N$.

Figure 11: Example 5.



As Figure11 above shows, observe that

$$w(\{1,2,3\}) = 2,$$
$$w(\{1,4\}) = w(\{2,5\}) = w\{3,6\} = 1.$$

Crucially, because $w(\{4,5,6\}) = 0$, by Lemma 2 (ii), no partition of $\{4,5,6\}$ can secure a positive payoff. Therefore, the Sender's payoff is strictly greater under $\{\{1,4\},\{2,5\},\{3,6\}\}$ than any partition that contains $\{1,2,3\}$. In particular, the example demonstrates that, while it is optimal to group $\{1,2,3\}$ whenever the set of Receivers is given by $\{1,2,3\}\cup\tilde{N}$, where $\tilde{N} \subset \{4,5,6\}$, when the set of Receiver is in fact N, then it is no longer optimal to group $\{1,2,3\}$. □

The next example shows that the greedy algorithm fails to attain an optimal communication when Receivers are contentious and homogenous but $|\Theta| = 4$.

Example 6. Suppose $\Theta = \{\theta^1,\theta^2,\theta^3,\theta^4\}$, $N = \{1,2,3,4,5,6\}$ and

$$(t_i)_{i=1}^{|N|} = \left(\left(\theta^1,\frac{2}{3}\right),\left(\theta^2,\frac{2}{3}\right),\left(\theta^3,\frac{2}{3}\right),\left(\theta^4,\frac{2}{3}\right),\left(\theta^3,\frac{2}{3}\right),\left(\theta^4,\frac{2}{3}\right)\right).$$

Let $\mu_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$. Since for any distinct $i, j, k \in \{1,2,3,4\}$,

$$\operatorname{co}\left(B_i \cup B_j\right) = \left\{\mu \in \Delta\Theta : \mu\left(\theta_i\right) + \mu\left(\theta_j\right) \geq \frac{2}{3}\right\},$$
$$\operatorname{co}\left(B_i \cup B_j \cup B_k\right) = \left\{\mu \in \Delta\Theta : \mu\left(\theta_i\right) + \mu\left(\theta_j\right) + \mu\left(\theta_k\right) \geq \frac{2}{3}\right\},$$

it follows that

$$\mu_0 \in \operatorname{co}\left(B_1 \cup B_2\right), \operatorname{co}\left(B_1 \cup B_3 \cup B_4\right), \operatorname{co}\left(B_2 \cup B_3 \cup B_4\right)$$
$$\mu_0 \notin \operatorname{co}\left(B_3 \cup B_4\right), \operatorname{co}\left(B_1 \cup B_3\right), \operatorname{co}\left(B_1 \cup B_4\right), \operatorname{co}\left(B_2 \cup B_3\right), \operatorname{co}\left(B_2 \cup B_4\right).$$

Since $\{1,2\}$ is the only pair of Receivers that can secure a payoff of one, the greedy algorithm would yield a partition that contains $\{1,2\}$. Moreover, because $\operatorname{co}(B_3 \cup B_4 \cup B_5 \cup B_6) = \operatorname{co}(B_3 \cup B_4)$, it follows that $w(\{3,4,5,6\}) = 0$, which, in turn, means that any subset of $\{3,4,5,6\}$ cannot secure a strictly positive payoff. Together, these imply that the greedy algorithm would a payoff of one to the Sender. However, the Sender can strictly do better by instead partitioning the Receivers as $\{\{1,3,4\},\{2,5,6\}\}$ because this partition yields a payoff of two for the Sender. □

A.3.3 Multiple partitions

The following example demonstrates that the ability to adopt communication strategies over multiple partitions can strictly benefit the Sender by allowing her to guarantee that two Receivers take action $a = 1$ when a single partition could only guarantee one Receiver to take action $a = 1$.

Example 7. Suppose $N = \{1,2,3\}$, $\Theta = \{\theta^1, \theta^2, \theta^3\}$, $\mu_0 = (\frac{1}{5}, \frac{3}{5}, \frac{3}{5})$, $t_i = \theta_i$ for each $i \in \{1,2,3\}$, and $\gamma_{\theta^1} = \frac{2}{5}$, $\gamma_{\theta^2} = \frac{13}{20}$, and $\gamma_{\theta^3} = \frac{1}{4}$. Since $\gamma_{\theta^1} + \gamma_{\theta^2} > 1$, $w(N) \leq 1$. However, we will show that if the Sender can partition the Receivers in multiple ways, she can guarantee a payoff of two. Specifically, suppose $\mathscr{P}_1 = \{\{1,2\},\{3\}\}$ and $\mathscr{P}_2 = \{\{1\},\{2,3\}\}$ and let $m_1 \in \{\ell_1, r_1\}$ be the message that the Sender sends to the group $\{1,2\}$ under $\mathscr{P}_1$ and $m_2 \in \{\ell_2, r_2\}$ be the message that the Sender sends to group $\{2,3\}$ under $\mathscr{P}_3$. Thus, Receiver 1 observes message m_1, Receiver 3 observes message m_2, and Receiver 2 observes message (m_1, m_2). Let $\pi : \Theta \to \Delta(\{\ell_1, r_1\} \times \{\ell_2, r_2\})$ be as shown in the table below.

Table 1: Example: Messaging strategy over multiple partition.

$\pi(m_1,m_2\|\theta)$	θ^1	θ^2	θ^3
(ℓ_1,ℓ_2)	0	0	0
(ℓ_1,r_2)	0	$\frac{3}{5}$	$\frac{4}{5}$
(r_1,ℓ_2)	0	$\frac{2}{5}$	0
(r_1,r_2)	1	0	$\frac{1}{5}$

The posterior beliefs for Receiver 1 who observes $m_1 \in \{\ell_1, r_1\}$ are

$$
\begin{aligned}
0 = \mu\left(\theta^1|\ell_1\right) &= \frac{\pi\left(\ell_1,r_2|\theta^1\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(\ell_1,r_2|\tilde{\theta}\right)}\\
&< \gamma_{\theta^1} = \frac{2}{5}\\
&\leq \frac{\pi\left(r_1,\ell_2|\theta^1\right)+\pi\left(r_1,r_2|\theta^1\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(r_1,\ell_2|\tilde{\theta}\right)+\pi\left(r_1,r_2|\tilde{\theta}\right)} = \mu\left(\theta^1|r_1\right) = \frac{5}{12}
\end{aligned}
$$

so that Receiver 1 takes action $a = 1$ if and only if $m_1 = r_1$. The posterior beliefs for Receiver 3 who observes $m_2 \in \{\ell_2, r_2\}$ are:

$$
\begin{aligned}
0 = \mu\left(\theta^3|\ell_2\right) &= \frac{\pi\left(r_1,\ell_2|\theta^3\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(r_1,\ell_2|\tilde{\theta}\right)}\\
&< \gamma_{\theta^3} = \frac{1}{4}\\
&\leq \frac{\pi\left(\ell_1,r_2|\theta^3\right)+\pi\left(r_1,r_2|\theta^3\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(\ell_1,r_2|\tilde{\theta}\right)+\pi\left(r_1,r_2|\tilde{\theta}\right)} = \mu\left(\theta^3|r_2\right) = \frac{5}{19}
\end{aligned}
$$

so that Receiver 3 takes action $a = 1$ if and only if $m_2 = r_2$. Posterior beliefs for Receiver 2 who observes $(m_1, m_2) \in \{\ell_1, r_1\} \times \{\ell_2, r_2\}$ are

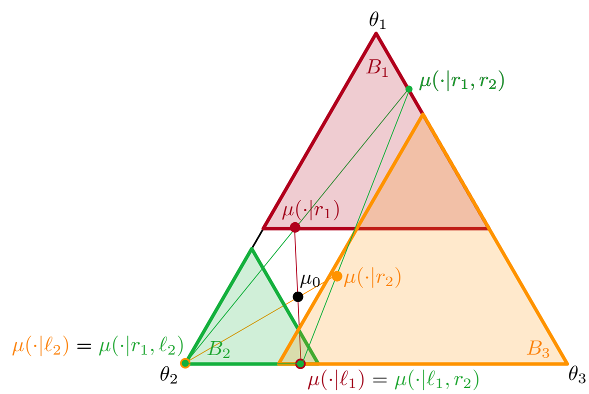
$$
\begin{aligned}
0 = \mu\left(\theta^2|r_1,r_2\right) &= \frac{\pi\left(r_1,r_2|\theta^2\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(r_1,r_2|\tilde{\theta}\right)}\\
&< \gamma_{\theta^2} = \frac{13}{20}\\
&< \mu\left(\theta^2|\ell_1,r_2\right) = \frac{\pi\left(\ell_1,r_2|\theta^2\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(\ell_1,r_2|\tilde{\theta}\right)} = \frac{9}{13}\\
&< \mu\left(\theta^2|r_1,\ell_2\right) = \frac{\pi\left(r_1,\ell_2|\theta^2\right)}{\sum_{\tilde{\theta}\in\Theta}\pi\left(r_1,\ell_2|\tilde{\theta}\right)} = 1
\end{aligned}
$$

so that Receiver 2 takes action $a = 1$ if and only if $(m_1, m_2) \in \{(r_1, \ell_2), \{\ell_1, r_2\}\}$. Therefore,

- $\triangleright$ if $(m_1, m_2) = (\ell_1, r_2)$, Receivers 2 and 3 take action $a = 1$;
- $\triangleright$ if $(m_1, m_2) = (\ell_1, r_2)$, Receivers 1 and 2 take action $a = 1$;
- $\triangleright$ if $(m_1, m_2) = (r_1, r_2)$, Receivers 1 and 3 take action $a = 1$.

Hence, the Sender has no incentive to deviate from π and she can guarantee a payoff of two from the group of two.

Figure 12: Example: Sender strictly benefits from multiple partitions.



$\square$